THE GROTHENDIECK GROUP OF ALGEBRAIC STACKS

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ABSTRACT. We introduce a Grothendieck group of algebraic stacks (with affine stabilisers) analogous to the Grothendieck group of algebraic varieties. We then identify it with a certain localisation of the Grothendieck group of algebraic varieties. Several invariants of elements in this group are discussed. The most important is an extension of the Euler characteristic (of cohomology with compact support) but in characteristic zero we introduce invariants which are able to distinguish between classes with the same Euler characteristic. These invariants are actually defined on the completed localised Grothendieck ring of varieties used in motivic integration. In particular we show that there are PSL_n-torsors of varieties whose class in the completed localised Grothendieck ring of varieties is not the product of the class of the base and the class of PSL_n.

Counting points of varieties over finite fields is both a well-established heuristic method to guess more precise (cohomological) statements as well as a way to obtain actual information on the ℓ-adic Galois representation Euler characteristics. Particularly in the latter case what has often been considered (starting at least with [HN74]) are moduli varieties (or usually more properly stacks) where one when counting constantly divides by the order of some automorphism group. There is a more direct approach to the cohomology of the moduli spaces considered by Harder-Narasimhan given by Atiyah-Bott ([AB83]). Though completely different in basic approach it is surprisingly similar in the logic of the recursion which inductively analyses the strata of a stratification. The main difference in execution is that point counting allows one to just sum up over the strata and divide by the order of a group when one is taking quotients by it. That one can sum over the strata is essentially because one is computing Euler characteristics of cohomology instead of getting hold of the actual cohomology (though a clever part of the Harder-Narasimhan argument is that one can compute the Betti numbers from the Galois representation Euler characteristic in many cases by using Deligne's fundamental weight result). The second part is somewhat different as the division by the group order corresponds to a spectral sequence involving equivariant cohomology. As this spectral sequence will in general have infinite total dimension it is not possible to directly compute Euler characteristics from it.

This article started off as an attempt to set things up so that the geometrical aspects of the point counting of [Be06] could be made more visible. For that one might want to go further than just making a cohomological argument as the geometric reasons are usually even of a "motivic" character.[1] Point counting of varieties can often be made motivic by working in a Grothendieck ring of varieties and the kind of point counting that is involved here should be made motivic by working in a Grothendieck ring of algebraic stacks. In such a ring the classes of many algebraic groups will be invertible so that one may indeed divide by them. I eventually found out that this idea had already appeared to several people (cf., [BD07, Jo07, To05]). I believe however that we manage to navigate the Scylla and Charibdis of these precedents to the point that the introduction of a new Grothendieck group of algebraic stacks can be argued (see the remark after Theorem 1.2 for an extended discussion of the relations between the present approach and these others). A more precise description of the contents is as follows.

The article starts with introducing a Grothendieck ring $K_0(\mathrm{Stck}_\mathbf{k})$ of algebraic stacks (and some natural quotients of it) over a field $\mathbf{k}$ and gives a number of situations when fibrations are multiplicative with respect to the classes of the stacks in this ring. It is then shown that this Grothendieck ring is actually a specific localisation of the Grothendieck ring $K_0(\mathrm{Spc}_\mathbf{k})$ of algebraic varieties. This localisation result is then used to give an Euler characteristic of any algebraic stack (of finite type and with affine stabilisers) in a suitably completed Grothendieck

[1]I use quotes as I do not use "motivic" in any technically precise way.

ring of mixed Galois representations or mixed Hodge structures. It is further shown that the image lies in a subring of this completion of elements of "polynomial growth". This subring is given a stronger topology than the completion topology. The main point about this stronger topology is that over a finite field the trace of the Frobenius on it is well-defined and continuous. It is then shown that this trace applied to the class of an algebraic stack is equal to the point count of the stack. Inspired by [BD07] these results are extended to certain algebraic stacks only locally of finite type. The Euler characteristic (for cohomology with compact support) has been defined by extension from the spatial case but we give the expected comparison with the recent definition of Laszlo and Olsson ([OL08b]) of cohomology with compact support for algebraic stacks.

Even though the point counting focusses the attention on the appropriate Euler characteristic of classes in $K_0(\mathrm{Spc_k})$ and its variants there are other invariants such as those of [LL04] and [Po02] that detect elements not detected by the Euler characteristic. The invariant of [LL04] does not extend to even $K_0(\mathrm{Spc_k})[\mathbb{L}^{-1}]$, where $\mathbb{L}$ is the class of $\mathbf{A}^1$, as it in fact vanishes on $\mathbb{L}$ and hence cannot be used for our purposes. Over the complex numbers we show how to extend Poonen's invariant to the completion of $K_0(\mathrm{Spc_k})$ and also define some other related invariants. We give two applications of these invariants. One is the generalisation of Poonen's example of zero-divisors in $K_0(\mathrm{Spc_k})$ to both its completion $\widehat{K}_0(\mathrm{Spc_k})$ and to $K_0(\mathrm{Stck_k})$. The other is showing that contrary to the case of GL_n-torsors we do not have multiplicativity for the classes of PSL_n-torsors in $K_0(\mathrm{Spc_k})$ and its variants. This is pertinent, in the case $G = \mathrm{PSL}_n$, to the question posed by Behrend and Dhillon (cf., [BD07, Rmk 2.10]) on the relation between $\widehat{K}_0(\mathrm{Spc_k})$ and $\widehat{K}_0^G(\mathrm{Spc_k})$. A more substantial use of these invariants will however be made in [Ek09b] where we shall show the non-triviality of the classes of the classifying stacks of certain finite groups.

At the end we make a sample computation for the class of the algebraic stack of smooth proper genus 1 curves with a polarisation of degree 3.

1   Stacks vs spaces

Recall that $K_0(\mathrm{Spc_k})$ is the Grothendieck group spanned by classes $\{X\}$ of algebraic $\mathbf{k}$-spaces X of finite type with the relations that $\{X\}$ only depends on the isomorphism class of X and $\{X\} = \{Y\} + \{U\}$ where Y is a closed subscheme of X and U its complement. (Usually one considers only schemes but that gives the same group, our choice is motivated by the further consideration of algebraic stacks which are most closely related to algebraic spaces.) By analogy we let $K_0(\mathrm{Stck_k})$ be the Grothendieck group spanned by classes $\{\mathcal{X}\}$ of algebraic stacks $\mathcal{X}$ of finite type over the field $\mathbf{k}$ all of whose automorphism group schemes are affine (we shall assume that the automorphism group schemes of all our stacks of this type without further mention) with relations

- $\{\mathcal{X}\}$ depends only on the isomorphism class of $\mathcal{X}$,

- $\{\mathcal{X}\} = \{\mathcal{Y}\} + \{\mathcal{U}\}$, where $\mathcal{Y}$ is a closed substack of $\mathcal{X}$ and $\mathcal{U}$ its complement and

- $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbf{A}^n\}$, where $\mathcal{E} \to \mathcal{X}$ is a vector bundle of constant rank n.[2]

As in the case of $K_0(\mathrm{Spc_k})$ we have a ring structure on $K_0(\mathrm{Stck_k})$ given by $\{\mathcal{X}\}\{\mathcal{Y}\} = \{\mathcal{X} \times \mathcal{Y}\}$. Putting $\mathbb{L} := \{\mathbf{A}^1\}$, the class of the affine line, the last condition above can then be rewritten as $\{\mathcal{E}\} = \mathbb{L}^n\{\mathcal{X}\}$. If $\mathcal{G}$ is a class of connected $\mathbf{k}$-group schemes of finite type, then we define $K_0^{\mathcal{G}}(\mathrm{Stck_k})$ as the quotient of $K_0(\mathrm{Stck_k})$ by the relations $\{Y\} = \{G\}\{X\}$ for every $G \in \mathcal{G}$ and every G-torsor $X \to Y$, where X and Y are *spaces*. We have a unique ring structure on $K_0^{\mathcal{G}}(\mathrm{Stck_k})$ making the quotient map $K_0(\mathrm{Stck_k}) \to K_0^{\mathcal{G}}(\mathrm{Stck_k})$ a ring homomorphism. Note also that if $\mathcal{G} \subseteq \mathcal{G}'$, then the quotient map $K_0(\mathrm{Stck_k}) \to K_0^{\mathcal{G}'}(\mathrm{Stck_k})$ factors through $K_0(\mathrm{Stck_k}) \to$

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[2]In the case of $K_0(\mathrm{Spc_k})$ this relation follows from the first two.

$K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$. By considering an algebraic space as an algebraic stack we of course also get a natural map $K_0(\mathrm{Spc}_{\mathbf{k}}) \to K_0(\mathrm{Stck}_{\mathbf{k}})$ as well as maps $K_0^{\mathcal{G}}(\mathrm{Spc}_{\mathbf{k}}) \to K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$ with the obvious definition of $K_0^{\mathcal{G}}(\mathrm{Spc}_{\mathbf{k}})$. We start by collecting some basic properties of $K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$ in the following proposition.

Proposition 1.1 *i) We have* $\{\mathrm{GL}_n\} = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$.

ii) If $\mathcal{X} \to \mathcal{Y}$ is a GL_n-torsor of algebraic $\mathbf{k}$-stacks of finite type, then $\{\mathcal{X}\} = \{\mathrm{GL}_n\}\{\mathcal{Y}\} \in K_0(\mathrm{Stck}_{\mathbf{k}})$.

iii) If $G \in \mathcal{G}$ and $\mathcal{X} \to \mathcal{Y}$ is a G-torsor of algebraic $\mathbf{k}$-stacks of finite type, then $\{\mathcal{X}\} = \{G\}\{\mathcal{Y}\} \in K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$.

iv) If, $H, N \in \mathcal{G}$, $1 \to N \to G \to H \to 1$ is an extension of algebraic groups and $\mathcal{X} \to \mathcal{Y}$ is a G-torsor, then $\{\mathcal{X}\} = \{G\}\{\mathcal{Y}\}$ in $K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$.

v) If $G \in \mathcal{G}$ then $\{G\}\{\mathrm{B}G\} = 1 \in K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$.

vi) If $G \in \mathcal{G}$, F is a G-space, $\mathcal{X} \to \mathcal{Y}$ is a G-torsor of algebraic $\mathbf{k}$-stacks of finite type and $\mathcal{Z} \to \mathcal{Y}$ the F-fibration associated to $\mathcal{X} \to \mathcal{Y}$ and the G-action on F, then $\{\mathcal{Z}\} = \{F\}\{\mathcal{Y}\}$.

vii) If $G \in \mathcal{G}$, $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X}' \to \mathcal{Y}$ are G-torsors and $\mathcal{Z} \to \mathcal{Y}$ the stack of isomorphisms $\mathcal{X} \xrightarrow{\sim} \mathcal{X}'$, then $\{\mathcal{Z}\} = \{G\}\{\mathcal{Y}\}$ in $K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$.

viii) If $G \in \mathcal{G}$, F is a G-space, H is an algebraic group, $H \to G$ a morphism of algebraic groups, $\mathcal{X} \to \mathcal{Y}$ an H-torsor of algebraic $\mathbf{k}$-stacks of finite type and $\mathcal{Z} \to \mathcal{Y}$ the F-fibration associated to $\mathcal{X} \to \mathcal{Y}$ and the H-action on F (given by its G-action and the homomorphism $H \to G$), then $\{\mathcal{Z}\} = \{F\}\{\mathcal{Y}\}$.

ix) If $G \in \mathcal{G}$ and $H \hookrightarrow G$ is a subgroup scheme, then $\{\mathrm{B}H\} = \{G/H\}\{\mathrm{B}G\} \in K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$.

PROOF: To start with i) we note that GL_n is isomorphic to the space of bases of $\mathbf{A}^n$. The space of $k+1$ linearly independent vectors fibres over the space of k linearly independent vectors with total space $\mathbf{A}^n$ times the base minus a vector bundle of rank k. The fact that $\{E\} = \mathbb{L}^k\{X\}$ for a vector bundle $E \to X$ of rank k implies i) by induction.

As for ii) its proof is the relative version of the proof of i); we get a succession of fibrations $\mathcal{X} = \mathcal{X}_n \to \mathcal{X}_{n-1} \to \cdots \to \mathcal{Y}$, where $\mathcal{X}_k$ is the stack of k linearly independent vectors in the vector bundle associated to $\mathcal{X}$. Each $\mathcal{X}_{k+1} \to \mathcal{X}_k$ is then the complement of a vector subbundle in a vector bundle and we use repeatedly the third set of defining relations of $K_0(\mathrm{Stck}_{\mathbf{k}})$ to get that $\{\mathcal{X}\} = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})\{\mathcal{Y}\}$. We then use i) to conclude.

To prove iii), we start by noticing that $\{\mathrm{GL}_n\}$ is invertible in $K_0(\mathrm{Stck}_{\mathbf{k}})$ for all n. Indeed, $\mathbf{Spec\,k} \to \mathrm{BGL}_n$ is a GL_n-torsor and hence we have $1 = \{\mathrm{GL}_n\}\{\mathrm{BGL}_n\}$. In particular for a GL_n-torsor $\mathcal{Z} \to \mathcal{W}$ we may write the relation of ii) as $\{\mathcal{W}\} = \{\mathrm{BGL}_n\}\{\mathcal{Z}\}$.

Continuing, the formula to be proved is additive in decompositions of the base $\mathcal{Y}$ as the union of a closed substack and its complement. Hence we may assume that $\mathcal{Y}$ is a global quotient, i.e., there is GL_n-torsor $Y \to \mathcal{Y}$ such that Y is an algebraic space. Hence we have a 2-cartesian diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{Y},
\end{array}
$$

where $X \to Y$ is a G-torsor of algebraic spaces and $X \to \mathcal{X}$ and $Y \to \mathcal{Y}$ are GL_n-torsors. This gives $\{\mathcal{X}\} = \{\mathrm{BGL}_n\}\{X\} = \{\mathrm{BGL}_n\}\{G\}\{Y\} = \{G\}\{\mathcal{Y}\}$ which proves iii). Then, applying iii) to the N-torsor $\mathcal{X} \to \mathcal{X}/N$ and to the H-torsor $\mathcal{X}/N \to \mathcal{Y}$ we get $\{\mathcal{X}\} = \{N\}\{H\}\{\mathcal{Y}\}$ but $G \to H$ is an N-torsor so that $\{G\} = \{N\}\{H\}$ which gives iv). Further, v) then follows by applying iii) to the G-torsor $\mathbf{Spec\,k} \to \mathrm{B}G$.

Now, to prove vi), the F-fibration associated to $\mathcal{X} \to \mathcal{Y}$ is the quotient $F \times_G \mathcal{X}$. As the quotient map $F \times \mathcal{X} \to F \times_G \mathcal{X}$ is a G-torsor we get $\{F \times_G \mathcal{X}\} = \{\mathrm{B}G\}\{F\}\{\mathcal{X}\} = \{\mathrm{B}G\}\{F\}\{G\}\{\mathcal{Y}\} = \{F\}\{\mathcal{Y}\}$.

In vii), $\mathcal{Z}$ is the fibration associated to the $G \times G$-torsor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}' \to \mathcal{Y}$ and the $G \times G$-space $G \times G/G$, where $G \subseteq G \times G$ is the diagonal embedding. Hence, vii) follows from vi) and iv).

Turning to viii) we note that the associated F-fibration, $F \times_H \mathcal{X}$, is isomorphic to the F-fibration associated to the G-torsor $G \times_H \mathcal{X} \to \mathcal{Y}$. Hence, viii) follows from vi) applied to this G-torsor.

Finally, ix) follows directly from vi) applied to the H-torsor $\mathbf{Spec\,k} \to \mathrm{B}G$ and the G-space G/H, using that $\mathrm{B}H = G/H \times_G \mathrm{B}G$. $\qquad\square$

Using the results just obtained we can obtain a precise relation between $K_0(\mathrm{Spc}_\mathbf{k})$ and $K_0(\mathrm{Stck}_\mathbf{k})$. Indeed, we let $K_0^{\mathcal{G}}(\mathrm{Spc}_\mathbf{k})'$ be the ring obtained from $K_0^{\mathcal{G}}(\mathrm{Spc}_\mathbf{k})$ by inverting $\mathbb{L}$ as well as $(\mathbb{L}^n - 1)$ for all $n > 0$.

Theorem 1.2 *For any $\mathcal{G}$, the natural map $K_0^{\mathcal{G}}(\mathrm{Spc}_\mathbf{k}) \to K_0^{\mathcal{G}}(\mathrm{Stck}_\mathbf{k})$ induces an isomorphism $K_0^{\mathcal{G}}(\mathrm{Spc}_\mathbf{k})' \to K_0^{\mathcal{G}}(\mathrm{Stck}_\mathbf{k})$.*

PROOF: By Proposition 1.1 we have that $\{\mathrm{GL}_n\} = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$ is invertible in $K_0^{\mathcal{G}}(\mathrm{Stck}_\mathbf{k})$ which implies that we do indeed get a factorisation $K_0^{\mathcal{G}}(\mathrm{Spc}_\mathbf{k})' \to K_0^{\mathcal{G}}(\mathrm{Stck}_\mathbf{k})$ and we shall proceed by defining a map in the other direction. If an algebraic stack $\mathcal{X}$ (of finite type and with affine stabilisers) is a global quotient $\mathcal{X} = [X/\mathrm{GL}_n]$, then we would like to define $\{\mathcal{X}\}$ in $K_0^{\mathcal{G}}(\mathrm{Spc}_\mathbf{k})'$ as $\{X\}/\{\mathrm{GL}_n\}$. Our localisation ensures that $\{\mathrm{GL}_n\}$ is indeed invertible. If we present $\mathcal{X}$ in a second way as a global quotient $\mathcal{X} = [X'/\mathrm{GL}_{n'}]$ then we may construct a 2-cartesian diagram

$$
\begin{array}{ccc}
X'' & \longrightarrow & X \\
\downarrow & & \downarrow \\
X' & \longrightarrow & \mathcal{X}
\end{array}
$$

and as $X'' \to X$ is a $\mathrm{GL}_{n'}$-torsor and $X'' \to X'$ a GL_n-torsor we get $\{\mathrm{GL}_{n'}\}\{X\} = \{X''\} = \{\mathrm{GL}_n\}\{X'\}$ which gives the independence. In the general case we stratify a stack $\mathcal{X}$ by global quotients and sum up. To show well-definedness we may assume that $\mathcal{X}$ is a global quotient and we get what we want by the independence just proved. $\qquad\square$

Remark: As mentioned in the introduction this theorem is already implicitly, though it seems not explicitly, in the literature. In [BD07] the authors associate to certain algebraic stacks (including those of finite type with affine stabilisers) an element in $\widehat{K}_0(\mathrm{Coh}_\mathbf{k})$. An analysis of the proof shows that if one restricts oneself to finite type stacks then one need only invert $\mathbb{L}$ and $\mathbb{L}^n - 1$ in $K_0(\mathrm{Spc}_\mathbf{k})$ for the construction to work. In any case the crucial point, that $\{X\}\{\mathrm{GL}_n\}^{-1}$ only depends on the stack $[X/\mathrm{GL}_n]$ and not on the actual action of GL_n on the algebraic space X is to be found in [BD07]. This is also [Jo07, Prop. 4.8] (though formally only in $K_0(\mathrm{Stck}_\mathbf{k}) \bigotimes \mathbf{Q}$). Finally, one can obtain the map $K_0(\mathrm{Stck}_\mathbf{k}) \to K_0(\mathrm{Spc}_\mathbf{k})'$ by including algebraic stacks in the larger category of n-stacks and using Toën's n-analogue, [To05, Thm 3.10], of the theorem. However, it seems worthwhile to give an explicit definition of a Grothendieck group of algebraic stacks whose relations are simpler than a restriction of the relations of Toën to the case of 1-stacks. Also giving a direct proof of the theorem is arguably more convenient for the reader than a piecing together of the arguments found in [BD07] and [Jo07]. Note also that parts of Proposition 1.1 are also implicitly in the quoted sources (or analogues in the case of [BD07]).

It is not at all clear how drastic the passage from $K_0(\mathrm{Spc}_\mathbf{k})$ to $K_0(\mathrm{Spc}_\mathbf{k})'$ is, we don't for instance know anything about the kernel of the localisation map $K_0(\mathrm{Spc}_\mathbf{k}) \to K_0(\mathrm{Spc}_\mathbf{k})'$. The following proposition gives a small amount of information.

Proposition 1.3 *Suppose that $\varphi(x)$ and $\rho(x)$ are integer polynomials. If $\varphi(\mathbb{L})$ divides $\rho(\mathbb{L})$ in $K_0(\mathrm{Spc}_\mathbf{k})$, then $\varphi(x)$ divides $\rho(x)$. In particular the only integer polynomials in $\mathbb{L}$ that are invertible in $K_0(\mathrm{Spc}_\mathbf{k})$ resp. $K_0(\mathrm{Stck}_\mathbf{k})$ are ± 1 resp. products of $\mathbb{L}$ and cyclotomic polynomials.*

PROOF: Assume that $\rho(\mathbb{L}) = \varphi(\mathbb{L})x$ for some $x \in K_0(\mathrm{Spc}_\mathbf{k})$. Applying χ_c (whose definition is recalled below) we get an equation $\rho(q) = \varphi(q)y$ in $K_0(\mathrm{Coh}_\mathbf{k})$. We now argue by induction on the degree of ρ. Let y' be the term of highest weight part of y and $a_n q^n$ the highest degree

term of $\varphi(q)$. Then $a_n q^n y'$ is the highest weight part of $\rho(q)$ and is hence of the form $b_m q^m$, where ρ is of degree m. As q is invertible we get $a_n q^{n-m} y' = b_m$. As $K_0(\mathrm{Coh}_{\mathbf{k}})$ is a free abelian group (free on the irreducible Galois representations or rational Hodge structures) we get that a_n divides b_m. Thus we get $\rho - (a_n/b_m)q^{n-m}\varphi(q) = \varphi(q)(y - (a_n/b_m)q^{n-m})$ and the induction assumption implies that φ divides ρ. The rest of the proposition follows from the first part (using that $K_0(\mathrm{Stck}_{\mathbf{k}}) = K_0(\mathrm{Spc}_{\mathbf{k}})'$). □

Now, let $\mathcal{Z}\mathrm{ar} = \mathcal{Z}\mathrm{ar}_{\mathbf{k}}$ be the class of connected finite type group schemes for which torsors over any (finitely generated) extension field of $\mathbf{k}$ are trivial. We have of course that $\mathrm{GL}_n, \mathrm{SL}_n \in \mathcal{Z}\mathrm{ar}$ but the following proposition provides more examples.

Proposition 1.4 *i) We have that $K_0(\mathrm{Stck}_{\mathbf{k}}) = K_0^{\mathcal{Z}\mathrm{ar}}(\mathrm{Stck}_{\mathbf{k}})$.*

ii) Let L be a finite dimensional $\mathbf{k}$-algebra. Then L^*, the group scheme of invertible elements of L, belongs to $\mathcal{Z}\mathrm{ar}$.

iii) $\mathcal{Z}\mathrm{ar}$ is closed under extensions.

PROOF: For i) using Theorem 1.2 it is enough to show that for any $G \in \mathcal{Z}\mathrm{ar}$ and any G-torsor $X \to Y$ of algebraic $\mathbf{k}$-spaces of finite type we have that $\{X\} = \{G\}\{Y\} \in K_0(\mathrm{Spc}_{\mathbf{k}})$. It is enough to find a stratification of Y such that the torsor is trivial over any stratum. We do this by Noetherian induction using that over each generic point of Y the torsor is trivial by assumption and the trivialisation can be extended to an open subset of Y.

For a $\mathbf{k}$-field K, L^*-torsors over K correspond to L_K-modules locally free of rank 1 and such modules are indeed free which proves ii).

The fact that $\mathcal{Z}\mathrm{ar}$ is closed under extensions is clear. □

Remark: Over an algebraically closed field the groups in $\mathcal{Z}\mathrm{ar}$ are precisely the special groups in the sense of [SC58, Exp. 1]. Indeed, a slight modification of the proof of [loc. cit., Thm 1] shows that a group $G \in \mathcal{Z}\mathrm{ar}$ is affine (and connected) and to show that it is special it suffices by [loc. cit., Thm 2] to show that $\mathrm{GL}_n \to \mathrm{GL}_n/G$ is Zariski locally trivial but as observed after that theorem this follows from the existence of a rational section which in turn it has as $G \in \mathcal{Z}\mathrm{ar}$. The third part of the proposition is, again in the algebraically closed case, [loc. cit., Lemme 6] while the second is in [Jo07]. Note that while the proof of these results in the non-algebraically closed case are the same as for the algebraically closed case some care has to be taken. For instance [loc. cit., Prop. 14], that solvable connected groups are special, is not true in general. There are in fact tori over suitable fields that are not special (or in $\mathcal{Z}\mathrm{ar}$) and over non-perfect fields there are also non-special connected unipotent algebraic groups.

2 Euler characteristics and point counting

We start by recalling the definition of the Euler characteristic of elements of $K_0(\mathrm{Spc}_{\mathbf{k}})$ and $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$. Recall that the Euler characteristic of cohomology with compact support is additive over a decomposition of an algebraic space into a closed subspace and its complement. This means that this Euler characteristic gives a map from $K_0(\mathrm{Spc}_{\mathbf{k}})$. We want the recipient group to be an appropriate Grothendieck group of cohomology groups with extra structure. When $\mathbf{k} = \mathbf{C}$ one such choice is the Grothendieck group of (polarisable) mixed Hodge structures (and when $\mathbf{k} = \mathbf{R}$ one could consider mixed Hodge structures with a complex conjugation). For a general field we would like to use actions of Galois groups of finitely generated subfields of $\mathbf{k}$. To be able to get the needed functoriality as well as to have a notion of mixedness we need to spread out even further. Hence we shall be considering the category that is the direct limit of the category of local $\mathbf{Q}_\ell$-systems on (spectra of) finitely generated subrings of $\mathbf{k}$. In concrete Galois-theoretic terms this means the following: Fixing an algebraic closure $\bar{\mathbf{k}}$ of $\mathbf{k}$ we can consider, for each finitely generated subring S of $\mathbf{k}$, the fundamental group $\pi_1(\mathbf{Spec}\, S, \bar{s})$, where $\bar{s}$ is the composite $\mathbf{Spec}\, \bar{\mathbf{k}} \to \mathbf{Spec}\, \mathbf{k} \to \mathbf{Spec}\, S$. An object in the category is then a continuous finite-dimensional $\mathbf{Q}_\ell$-representation of $\pi_1(\mathbf{Spec}\, S, \bar{s})$ for some S. A morphism between two such

representations associated to R and S is then a choice of a T containing both R and S and a map between the induced $\pi_1(\mathbf{Spec}\,T, t)$-representations. Such a representation is *mixed* if, after possibly extending S, it is a successive extension of pure representations. A representation is *pure* (of weight n) if for every closed point x of $\mathbf{Spec}\,S$, the eigenvalues of the (geometric) Frobenius map associated to x are algebraic numbers all of whose (archimedean) absolute values are equal to $q^{n/2}$, where $q = |\mathbf{k}(x)|$. Note that when S is normal (which can be insured by extending S) the map $\mathrm{Gal}(\overline{K}/K) \to \pi_1(\mathbf{Spec}\,S, s)$, where K is the fraction field of S and $\overline{K}$ the algebraic closure of K in $\bar{\mathbf{k}}$, is surjective. Hence $\pi_1(\mathbf{Spec}\,S, s)$-representations can be thought of as $\mathrm{Gal}(\overline{K}/K)$-representations for which the kernel of $\mathrm{Gal}(\overline{K}/K) \to \pi_1(\mathbf{Spec}\,S, s)$ acts trivially. In this way our objects can indeed be thought of as Galois representations for finitely generated subfields of $\mathbf{k}$. When $\mathbf{k}$ is algebraically closed things can be made even more concrete: One chooses a finitely generated subring S of $\mathbf{k}$ and considers representations of open subgroups of $\pi_1(\mathbf{Spec}\,S, \overline{s})$ where morphisms are only required to commute with the elements of some open subgroup. In the general case one does not allow all open subgroups but only some. If one does not wish to take this direct limit of categories of mixed representations over finitely generated subfields one may instead define $K_0(\mathrm{Coh}_{\mathbf{k}})$ as follows:

- If $\mathbf{k}$ is a finitely generated field, then $K_0(\mathrm{Coh}_{\mathbf{k}})$ is the Grothendieck group of the category of mixed $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$-representations, i.e., $\mathbf{Q}_\ell$-representations $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}) \to \mathrm{GL}(V)$ which factor through a quotient $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}) \to \pi_1(S, \bar{\mathbf{k}})$, where S is a normal integral scheme with $\mathbf{k}$ as function field of finite type over $\mathbf{Z}$ and which are mixed. This group does not depend on $\bar{\mathbf{k}}$ and is functorial for field inclusions.

- For a general field $K_0(\mathrm{Coh}_{\mathbf{k}}) := \varinjlim_K K_0(\mathrm{Coh}_K)$, where K runs over finitely generated subfields of $\mathbf{k}$.

In any case we shall use $K_0(\mathrm{Coh}_{\mathbf{k}})$ to denote the Grothendieck group of either mixed Galois $\mathbf{Q}_\ell$-representations as above or when $\mathbf{k} = \mathbf{C}$ ($\mathbf{k} = \mathbf{R}$) the mixed polarisable Hodge structures (resp. those provided with a complex conjugation). The additivity of the Euler characteristic (of cohomology with compact support) can then be formulated as saying that it gives a ring homomorphism $\chi_c \colon K_0(\mathrm{Spc}_{\mathbf{k}}) \to K_0(\mathrm{Coh}_{\mathbf{k}})$ (where the ring structure on $K_0(\mathrm{Coh}_{\mathbf{k}})$ is induced by the tensor product of representations). Note also that the Euler characteristic is multiplicative for torsors over a connected group scheme (indeed all higher direct images of a constant sheaf are constant as they become constant after a base change with connected fibres) so that χ_c factors over a $\chi_c \colon K_0^{\mathcal{G}}(\mathrm{Spc}_{\mathbf{k}}) \to K_0(\mathrm{Coh}_{\mathbf{k}})$ for an arbitrary $\mathcal{G}$.

Now, $\chi_c(\mathbb{L})$ is invertible but $\chi_c(\mathbb{L}^n - 1)$ is not. In order for us to be able to use Theorem 1.2 to extend χ_c to $K_0(\mathrm{Stck}_{\mathbf{k}})$ we could of course just invert the $\chi_c(\mathbb{L}^n - 1)$. However, we shall follow the established road of making a completion instead. For this note that as every cohomology object has finite length and the simple objects are pure of some weight, $K_0(\mathrm{Coh}_{\mathbf{k}})$ is graded by weight. Hence, the completion, which completes with respect to the weight filtration in the direction of negative weights is very benign in that it just replaces finite sums with sums unbounded in the negative direction. We shall use $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$ to denote this completion. Putting $q := \chi_c(\mathbb{L})$ we then have $\chi_c(\mathbb{L}^n - 1) = q^n - 1$ which is invertible with inverse $q^{-n} + q^{-2n} + \cdots$ so that we do indeed get an extension $\chi_c \colon K_0(\mathrm{Stck}_{\mathbf{k}}) \to \widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$.

We now want to compare this with the process of point counting. Hence, if $\mathbf{F}$ is a finite field we have a map $|-| \colon K_0(\mathrm{Stck}_{\mathbf{F}}) \to \mathbf{Q}$ by associating to an algebraic stack (of finite type) $\mathcal{X}$ the mass of the essentially finite groupoid $\mathcal{X}(\mathbf{F})$ (the *mass* of an essentially finite groupoid is the sum over its isomorphism classes of 1 over the order of the automorphism groups). On the other hand, for an element $\sum_{i \leq N} a_i$ of $\widehat{K}_0(\mathrm{Coh}_{\mathbf{F}})$ (with a_i pure of weight i) one can consider the sum $\sum_{i \leq N} \mathrm{Tr}(a_i)$ where $\mathrm{Tr} \colon \widehat{K}_0(\mathrm{Coh}_{\mathbf{F}}) \to \bar{\mathbf{Q}}$ is the map that takes a Galois representation to the trace of the action of the Frobenius element (where we have fixed once and for all an embedding of the algebraic numbers of $\mathbf{Q}_\ell$ into $\bar{\mathbf{Q}}$). In general this sum will not be convergent and one needs to add some conditions on a sum in order to be allowed to take its trace. In [BD07] the authors use what seems essentially to be the weakest possible condition namely that the sum is absolutely

convergent. However, this seems to me to be too *ad hoc*, for instance possibly depending on the precise finite field that one is considering and also makes sense only when the base field is finite. I instead propose the following approach:

Given a k-scheme X of finite type and an integer n we define $w_n(X)$ to be $\sum_i w_n(H^i_c(X))$, where $w_n(H^i_c(X))$ is the dimension of the part of $H^i_c(X)$ of weight n (note that in the case when $\mathbf{k}$ equals $\mathbf{C}$ or $\mathbf{R}$ this is not ambiguous as the weight filtration on the classical cohomology is compatible with that on étale cohomology). We then define, for $x \in K_0(\mathrm{Spc}_{\mathbf{k}})$ $w'_n(x)$ to be the minimum of $\sum_k w_n(X_k)$ where $\sum_k \pm\{X_k\}$ runs over all representatives of x as a signed sum of classes of schemes. Note that $w_{n+2}(X \times \mathbf{A}^1) = w_n(X)$ and thus $w'_{n+2}(x\mathbb{L}) \leq w'_n(x)$. This means that we may put $\overline{w}_n(x) := \lim_{k \to \infty} w'_{n+2k}(x\mathbb{L}^k)$. We now extend $\overline{w}_n$ to $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ by putting $\overline{w}_n(x\mathbb{L}^{-k}) := \overline{w}_{n+2k}(x)$ (this is consistent as by construction $\overline{w}_{n+2}(x\mathbb{L}) = \overline{w}_n(x)$ for $x \in K_0(\mathrm{Spc}_{\mathbf{k}})$). We clearly have subadditivity for $\overline{w}_n$, $\overline{w}_n(x \pm y) \leq \overline{w}_n(x) + \overline{w}_n(y)$, as well as submultiplicativity, $\overline{w}_n(xy) \leq \sum_{i+j=n} \overline{w}_i(x)\overline{w}_j(y)$, by the Künneth formula. The next step is to complete $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ by adding equivalence classes of Cauchy sequences $x = (x_i)$ fulfilling two conditions. The first uses the dimension filtration $\{\mathrm{Fil}^{\leq m} K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]\}$ where $\mathrm{Fil}^{\leq m} K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ is spanned by classes $\{X\}\mathbb{L}^{-k}$ with $\dim(X) - k \leq m$ and $\dim(x)$ is the least (possibly equal to $-\infty$) number such that $x \in \mathrm{Fil}^{\leq m} K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$. We then, to begin with, demand that $\dim(x_i - x_j) \to -\infty$ when $i, j \to \infty$. We further impose that there be constants C and D both independent of i such that $\overline{w}_n(x_i) \leq C|n|^d + D$ for all n. We shall say that such a sequence is *convergent of uniform polynomial growth*. Note that by results of Deligne (cf., [De80, De74]) we have that $\overline{w}_n(x) = 0$ if $2\dim(x) < n$ and thus (by subadditivity) for any n, $\overline{w}_n(x_i)$ is eventually constant and we may unambigously define $\overline{w}_n(x)$ as that constant value. As (x_i) is of uniformly polynomial growth we also get that $\overline{w}_n(x)$ is of polynomial growth.

The condition of polynomial growth can be conveniently formulated in the following way. For $x \in K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ we put $\overline{w}(x, t) = \sum_n \overline{w}_n(x)t^n$ considered as an element of $\mathbf{Z}((t^{-1}))$, Laurent series in t^{-1} with integer coefficients. That a sequence (x_i) is of uniform means that $\overline{w}(x_i, t) \leq \varphi(t)/(t-1)^d$ for some Laurent polynomial φ and an integer d both independent of i and where comparison is made coefficient-wise. Furthermore, subadditivity can be formulated as $\overline{w}(x \pm y, t) \leq \overline{w}(x, t) + \overline{w}(x, t)$ and submultiplicativity as $\overline{w}(xy, t) \leq \overline{w}(x, t)\overline{w}(y, t)$. These facts make it obvious that convergent sequences uniformly of polynomial growth are closed under term-wise addition, subtraction and multiplication. As those that converge to zero (wrt the dimension filtration) form an ideal we may define $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ as the quotient of the convergent sequences uniformly of polynomial growth by those converging to zero. We can extend the $\overline{w}_n$ (or equivalently $\overline{w}(-, t)$) to $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ and it is clear that convergent sequences uniformly of polynomial growth of elements of $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ converge in $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$. We of course have a completion map $\overline{(-)}\colon K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}] \to \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$. We also get a continuous map from $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ to the completion $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ of $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ in the topology given by the dimension filtration. By definition a uniformly convergent sequence converges to zero precisely when it converges to zero in the dimension filtration topology. Hence the map $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}}) \to \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ is injective and we may think of $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ as a subring of $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$. Its topology is however stronger than the subring topology.

In a similar manner we also define a subring $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$ of $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$ consisting of those elements x for which $\overline{w}_n(x)$ has at most polynomial growth. Here we define $\overline{w}_n(x)$ to be the sum $\sum_k |m_k|\dim V_k$, where $\sum_k m_k[V_k]$ is the weight n-component of x written as a sum of non-isomorphic irreducible Galois representations (resp. Hodge structures). It is clear that if $x \in K_0(\mathrm{Spc}_{\mathbf{k}})$, then $\overline{w}_n(\chi_c(x)) \leq \overline{w}_n(x)$ so that χ_c extends to a map $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}}) \to \overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$. We want however to give $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$ a stronger topology than that provided by $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$. Thus we declare a sequence (x_i) in $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$ be *convergent of uniform polynomial growth* if it is convergent in the filtration topology (i.e., for every n the weight n-components of the x_i is eventually constant) and $\overline{w}_n(x_i)$ is uniformly of polynomial growth as above. Then (x_i)

converges in $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$ to an element of $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$. (The topology is defined by having a subset being closed if it is closed under uniformly convergent sequences.) Given an element $\sum_n x_n$ in $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$, the partial sums $\sum_{n \geq -N} x_n$ converge uniformly to the element so that $K_0(\mathrm{Coh}_{\mathbf{k}})$ is dense in $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$. Furthermore, as $\overline{w}_n(\chi_c(x)) \leq \overline{w}_n(x)$ we get a continuous extension $\chi_c \colon \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}}) \to \overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$ which is a ring homomorphism.

Before our next result we need to recall that for an algebraic stack $\mathcal{X}$ locally of finite type over a finite field $\mathbf{k}$ we can define its *point count*, $\mathrm{cnt}(\mathcal{X}) \in [0, \infty]$, to be the mass of the groupoid $\mathcal{X}(\mathbf{k})$, i.e., the sum $\sum_x 1/|\mathrm{Aut}(x)|$, where the sum runs over the isomorphism classes of $\mathcal{X}(\mathbf{k})$ (and $\mathrm{Aut}(x)$ is the group of automorphisms *defined* over $\mathbf{k}$). Being a sum over non-negative numbers it is always convergent though it may be ∞; when $\mathcal{X}$ is of finite type it is a finite sum however. The point count is additive over a decomposition into a closed substack and its complement and hence gives a ring homomorphism $\mathrm{cnt} \colon K_0(\mathrm{Stck}_{\mathbf{k}}) \to \mathbf{Q}$. Note that its composite with the natural map $K_0(\mathrm{Spc}_{\mathbf{k}}) \to K_0(\mathrm{Stck}_{\mathbf{k}})$ is the usual counting map. Finally, we have a map $\mathrm{Tr} \colon K_0(\mathrm{Coh}_{\mathbf{k}}) \to \bar{\mathbf{Q}}$ given by associating to any Galois representation the trace of the Frobenius map and where we once and for all have chosen an embedding of $\bar{\mathbf{Q}}$, the field of complex algebraic numbers, in an algebraic closure of $\mathbf{Q}_\ell$. We now have the following (expected) compatibility result. For the second part we define, in analogy with the spatial case, the dimension filtration $\mathrm{Fil}^{\leq m} K_0(\mathrm{Stck}_{\mathbf{k}}) \subseteq K_0(\mathrm{Stck}_{\mathbf{k}})$ to be the subgroup spanned by the algebraic stacks of dimension $\leq m$. We also extend $\mathrm{Fil}^{\leq m} K_0(\mathrm{Spc}_{\mathbf{k}})$ to $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ by taking its closure.

Proposition 2.1 *i) The elements $\mathbb{L}$ and the $\mathbb{L}^n - 1$, all $n > 0$, of $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ are invertible so that the completion map extends to a ring homomorphism $K_0(\mathrm{Spc}_{\mathbf{k}})' \to \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ and thus to a map $K_0(\mathrm{Stck}_{\mathbf{k}}) \to \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ which shall also be denoted $\overline{(-)}$.*

ii) The completion map $\overline{(-)} \colon K_0(\mathrm{Stck}_{\mathbf{k}}) \to \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ takes $\mathrm{Fil}^{\leq m} K_0(\mathrm{Stck}_{\mathbf{k}})$ to $\mathrm{Fil}^{\leq m} \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$. In particular it is continuous for the filtration topologies.

Assume now that $\mathbf{k}$ is a finite field.

iii) For every $x = \sum_{n \leq N} x_n \in \overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$, where x_n is the component of weight n the sum $\sum_n \mathrm{Tr}(x_n)$ is convergent in $\mathbf{C}$ and gives a continuous extension of Tr to a ring homomorphism $\mathrm{Tr} \colon \overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}}) \to \mathbf{C}$.

iv) For every $x \in K_0(\mathrm{Stck}_{\mathbf{k}})$ we have that $\mathrm{cnt}(x) = \mathrm{Tr}(\chi_c(\overline{x}))$.

v) $\mathrm{cnt} \colon K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}] \to \mathbf{Q}$ extends to a continuous ring homomorphism $\mathrm{cnt} \colon \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}}) \to \mathbf{R}$.

PROOF: We have that $\mathbb{L}$ is invertible already in $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$. Furthermore, $\mathbb{L}^n - 1 = \mathbb{L}^n(1 - \mathbb{L}^{-n})$ and the series $\sum_{k \geq -1} \mathbb{L}^{-kn}$ is a convergent sum, uniformly of polynomial growth. This proves i).

To prove ii) we may consider only the case $x = \{\mathcal{X}\}$ where $\dim(\mathcal{X}) \leq m$ and by cutting up $\mathcal{X}$ we may further assume that $\mathcal{X} = [X/\mathrm{GL}_n]$ so that $\dim X \leq m + n^2$. We then have $\overline{\{\mathcal{X}\}} = \{X\}\{\mathrm{GL}_n\}^{-1}$ but $\{\mathrm{GL}_n\}^{-1}$ is a power series in $\mathbb{L}^{-1}$ starting with $\mathbb{L}^{-n^2}$ so that $\{X\}\{\mathrm{GL}_n\}^{-1} \in \mathrm{Fil}^{\dim X - n^2} \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}}) = \mathrm{Fil}^m \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$.

As for iii), if $\mathbf{k}$ has q elements then by definition there is a d such that $|\mathrm{Tr}(x_n)| \leq |n|^d q^{n/2}$ for all n which gives convergence as $q > 1$. Furthermore, if (x_i) is a convergent sequence of uniform polynomial growth converging in $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$ to x, then there are C, D and d such that $\overline{w}_n x_i \leq C|n|^d + D$ for all n and i and then

$$| \mathrm{Tr}(x) - \mathrm{Tr}(x_i)| \leq \sum_{k \leq 2 \dim(x - x_i)} 2(C|k|^d + D)q^{k/2},$$

which shows that as $\dim(x - x_i) \to -\infty$ as $i \to \infty$ we have $\mathrm{Tr}(x_i) \to \mathrm{Tr}(x)$.

Further, as both sides are ring homomorphisms, to verify the equality of iv), it is enough to check it on $K_0(\mathrm{Spc}_{\mathbf{k}})$ where it is simply the Lefschetz fixed point formula.

Finally, to prove v) we note that $\mathrm{Tr} \circ \chi_c$ is a continuous extension of cnt to $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc_k})$. $\qquad \square$

Remark: i) For the truth of the proposition it would be enough to have $\overline{w}_n(x)$ grow slower than α^{-n} for all $\alpha > 1$ (or even for some $\alpha < 2$). However, the requirement of polynomial growth seems to me to be the more natural condition (see however comment below).

ii) Note that $\overline{w}_n(\chi_c(x))$ (for $x \in K_0(\mathrm{Spc_k})$ say) is in general smaller than $\overline{w}_n(x)$ because of possible cancellations of irreducible Galois representations which do not correspond to cancellations of varieties in $K_0(\mathrm{Spc_k})$. Working with a Grothendieck group of motives instead of varieties would no doubt get a value of $\overline{w}_n$ closer to that of $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh_k})$. On the other hand staying in $K_0(\mathrm{Spc_k})$ gives more information. I also find it quite difficult to imagine that there are natural examples of elements x in the dimension completion of $K_0(\mathrm{Spc_k})$ for which $\overline{w}_n(x)$ does not have polynomial growth while $\overline{w}_n(\chi_c(x))$ does.

iii) The extension of χ_c to $K_0(\mathrm{Stck_k})$ and the relation with point counting for **k** a finite field is done in [To05] for the general case of n-stacks using the same idea.

2.1 Stacks of polynomial growth

The aim of this section is to try to fit the considerations of [BD07] into our setup. Hence, if $\mathcal{X}$ is an algebraic stack locally of finite type (and as usual with affine stabilisers), a *stratification* of $\mathcal{X}$ will mean a countable set $(\mathcal{X}_i)$ of locally closed substacks of finite type such that $\mathcal{X}$ is the disjoint union of them and $\dim(\mathcal{X}_i) \to -\infty$ as $i \to \infty$. (Note that we do *not* assume for instance that the closure of a stratum is a union of strata.) Following [BD07] we say that $\mathcal{X}$ is *essentially of finite type* if it has a stratification. The stratification is *uniformly of polynomial growth* if the sum $\sum_i \overline{\{\mathcal{X}_i\}}$ (we shall from now one dispense with the completion symbol) converges in $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc_k})$. In concrete terms this means that there is a polynomial $P(x)$ such that $\overline{w}_n(\sum_{i \in S} \mathcal{X}_i) \leq P(|n|)$ for all n and all finite subsets S of the index set as convergence in the filtration topology is assured by (2.1:ii). (This is rather the summability than the convergence series version but they are easily seen to be equivalent.) We say that $\mathcal{X}$ is of *polynomial growth* if it has some stratification $(\mathcal{X}_i)$ which is uniformly of polynomial growth. We then want to define $\{\mathcal{X}\} \in \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc_k})$ as $\sum\{\mathcal{X}_i\}$. This is indeed independent of the choice of stratification. To prove that, it is enough to show equality in $\widehat{K}_0(\mathrm{Spc_k})$ (as the map $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc_k}) \to \widehat{K}_0(\mathrm{Spc_k})$ is injective). Hence, if we simply assume that $(\mathcal{X}_i)$ and $(\mathcal{Y}_j)$ are exhaustions of $\mathcal{X}$ we see that both sums $\sum_i\{\mathcal{X}_i\}$ and $\sum_j\{\mathcal{Y}_j\}$ are equal to $\sum_{i,j}\{\mathcal{X}_i \cap \mathcal{Y}_j\}$ as for each i the stratification $(\mathcal{X}_i \cap \mathcal{Y}_j)$ of $\mathcal{X}_i$ is a finite stratification as $\mathcal{X}_i$ is of finite type (and similarly for the stratification $(\mathcal{Y}_j \cap \mathcal{X}_i)$ of $\mathcal{Y}_j$). This gives $\{\mathcal{X}_i\} = \sum_j\{\mathcal{X}_i \cap \mathcal{Y}_j\}$ and $\{\mathcal{Y}_j\} = \sum_i\{\mathcal{Y}_j \cap \mathcal{X}_i\}$. We then have a few basic properties.

Proposition 2.2 i) *Suppose that $\mathcal{X}$ is an algebraic **k**-stack locally of finite type and $\mathcal{Y}$ is a closed substack with complement $\mathcal{U}$. If the stacks $\mathcal{Y}$ and $\mathcal{U}$ are of polynomial growth then so is $\mathcal{X}$ and we have*

$$\{\mathcal{X}\} = \{\mathcal{Y}\} + \{\mathcal{U}\} \in \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc_k}).$$

ii) *If $\mathcal{X}$ is a stack of polynomial growth over a finite field **k**, then $\mathrm{Tr}(\chi_c\{\mathcal{X}\}) = \mathrm{cnt}(\mathcal{X})$.*

PROOF: Suppose that $\mathcal{Y}$ resp. $\mathcal{U}$ have stratifications $(\mathcal{Y}_i)$ resp. $(\mathcal{U}_i)$ of uniform polynomial growth. Their union then gives a stratification of $\mathcal{X}$ of uniform polynomial growth. This proves i).

As for ii) we have, by definition of the point count, that $\mathrm{cnt}(\mathcal{X}) = \sum_i \mathrm{cnt}(\mathcal{X}_i)$. Hence by Proposition 2.1 we have

$$\mathrm{Tr}(\{\mathcal{X}\}) = \sum_i \mathrm{Tr}(\{\mathcal{X}_i\}) = \sum_i \mathrm{cnt}(\mathcal{X}_i) = \mathrm{cnt}(\mathcal{X}).$$

$\qquad \square$

Remark: i) It would seem reasonable to demand a version for the first part that involves an infinite decomposition of $\mathcal{X}$ into disjoint substacks. However, one would to begin with have to

impose some uniformity on the dimensions of the strata of the stratification of the individual pieces. A second problem is that if $(\mathcal{X}_i)$ is a stratification of $\mathcal{X}$ then it is not clear that one can bound $\sum_i \overline{w}(\{\mathcal{X}_i\})$ in terms of $\overline{w}_n(\{\mathcal{X}\})$ which would be needed to bound $\sum_i \overline{w}_n(\{\mathcal{X}_i\})$ in terms of $\sum_{i,j} \overline{w}_n(\{\mathcal{X}_{i,j}\})$, where $(\mathcal{X}_{i,j})$ is a stratification of $\mathcal{X}_i$ of uniform polynomial growth (and $(\mathcal{X}_i)$ is a decomposition of $\mathcal{X}$).

ii) Another problem with our definition is that (even open) substacks of a stack of polynomial growth may not be of polynomial growth. This can be mitigated by modifying the notion of convergence. Unfortunately under that modification the Euler characteristic is no longer necessarily of polynomial growth.

We finish this subsection by verifying that the moduli stacks of [BD07], stacks of bundles over a smooth and proper curve for a split semi-simple group G.

Proposition 2.3 *Let G be a split semi-simple $\mathbf{k}$-group scheme and C a smooth and proper curve over C. Then the algebraic stack $\mathfrak{Bun}_{G,C}$ of G-torsors over C is of polynomial growth.*

PROOF: This is just going into the details of [BD07, Lemma 5.5] to see that the convergence is actually polynomial using [Be91, §8], particularly [Be91, §8.4] to get the requisite information. For the reader's (and the author's) convenience we shall only treat the notationally (and conceptually) simpler case of $G = \mathrm{SL}_n$. Hence we stratify $\mathfrak{Bun}_{\mathrm{SL}_n,C}$ by the strata of fixed ranks and degrees of the pieces of the Harder-Narasimhan filtration. Hence they are given by one sequence $(d_1, d_2, \ldots, d_k)$ of positive integers giving the ranks and one increasing sequence $(m_1, \ldots, m_k)$ of integers representing the degrees of the pieces of the filtration. That the whole vector bundle comes from an SL_n-bundle means that $\sum_i d_i = n$ and that we have a Harder-Narasimhan filtration means that $m_1/d_1 > m_2/d_2 > \cdots > m_k/d_k$. There is a small problem in that on a given stratum the Harder-Narasimhan filtration may not exist. However, the map from the algebraic stack of vector bundles with a Harder-Narasimhan filtration of the given type to the stratum is a universal homeomorphism. Generally, if $\mathcal{X} \to \mathcal{Y}$ is a universal homeomorphism between algebraic stacks of finite type we have that $\chi_c(\{\mathcal{X}\}) = \chi_c(\{\mathcal{Y}\})$ in $\overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}})$, a fact which follows immediately by reduction to the spatial case. Hence, we may and shall ignore this problem. Now, $\mathfrak{Bun}_C^{d,m}$ is the stack of flags $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k$ of vector bundles on C with $\mathcal{E}_i/\mathcal{E}_{i-1}$ semi-stable of rank d_i and degree m_i (and where we have $\sum_i d_i = n$ and $m_1/d_1 > m_2/d_2 > \cdots > m_k/d_k$). We now consider the map $\mathfrak{Bun}_C^{d,m} \to \mathfrak{Bun}_C^{d',m'} \times \mathfrak{Bun}_C^{d_k,m_k}$ mapping a flag to $(0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{k-1}, \mathcal{E}_k/\mathcal{E}_{k-1})$ and $d' = (d_1, \ldots, d_{k-1})$ and $m' = (m_1, \ldots, m_{k-1})$. What is lost under this map is the extension of $\mathcal{E}_{k-1}$ by $\mathcal{E}_k/\mathcal{E}_{k-1}$ given by $\mathcal{E}_k$. This makes $\mathfrak{Bun}_C^{d,m} \to \mathfrak{Bun}_C^{d',m'} \times \mathfrak{Bun}_C^{d_k,m_k}$ a Picard stack under the Baer sum of extensions which is described by the perfect complex $R\pi_*\mathcal{H}om(\mathcal{F}, \mathcal{E}_{k-1})[1]$, where $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{k-1}$ is the universal flag over $\mathfrak{Bun}_C^{d',m'} \times C$, $\mathcal{F}$ the universal vector bundle over $\mathfrak{Bun}_C^{d_k,m_k} \times C$ and π is the projection $\mathfrak{Bun}_C^{d',m'} \times \mathfrak{Bun}_C^{d_k,m_k} \times C \to \mathfrak{Bun}_C^{d',m'} \times \mathfrak{Bun}_C^{d_k,m_k}$. We may then stratify the base so that $\pi_*\mathcal{H}om(\mathcal{F}, \mathcal{E}_{k-1})$ is free of rank a, say, and $R^1\pi_*\mathcal{H}om(\mathcal{F}, \mathcal{E}_{k-1})$ is free of rank b, say, and so that $R\pi_*\mathcal{H}om(\mathcal{F}, \mathcal{E}_{k-1})$ is the sum of its cohomology. This gives that over such a stratum $\mathcal{X}$, π is isomorphic to $B\mathbf{G}_a^a \times \mathbf{G}_a^b \times$. Using additivity of $\{-\}$ we get

$$\{\mathfrak{Bun}_C^{d,m}\} = \mathbb{L}^{b-a}\{\mathfrak{Bun}_C^{d',m'} \times \mathfrak{Bun}_C^{d_k,m_k}\} = \mathbb{L}^{b-a}\{\mathfrak{Bun}_C^{d',m'}\}\{\mathfrak{Bun}_C^{d_k,m_k}\}$$

and $a - b$ is simply the (relative) Euler characteristic of $\mathcal{H}om(\mathcal{F}, \mathcal{E}_{k-1})$ which equals

$$\sum_{i<k} d_k m_i - d_i m_k$$

by the Riemann-Roch formula. Iterating we get

$$\{\mathfrak{Bun}_C^{d,m}\} = \mathbb{L}^{-\chi}\{\mathfrak{Bun}_C^{d_1,m_1}\} \cdots \{\mathfrak{Bun}_C^{d_k,m_k}\},$$

where

$$\chi = \sum_{i<j} d_i d_j \left(\frac{m_j}{d_j} - \frac{m_i}{d_i} \right) + d_i d_j (1 - g(C)).$$

Now, for fixed d $\sum_{i<j} d_i d_j (\frac{m_j}{d_j} - \frac{m_i}{d_i})$ is a linear function whose kernel clearly meets the cone given by the conditions $m_1/d_1 > m_2/d_2 \geq \cdots \geq m_k/d_k$ and $\sum_i m_i = 0$ only in 0. Hence the number of m's which give a fixed value of χ is polynomially bounded and only takes on a finite number of negative values. As also the isomorphism class of each $\mathfrak{Bun}_C^{d_i, m_i}$ only depends on the residue of m_i modulo d_i (provided that the field is large enough so that C has a line bundle of degree 1, field extensions are allowed however) we get that sum $\sum_{d,m} \{\mathfrak{Bun}_C^{d,m}\}$ is uniformly polynomially convergent. $\qquad \square$

Remark: The cohomology of the moduli spaces in question ultimately comes from the cohomology of the group in question, its loop space and its classifying space (cf., [AB83]). The polynomial growth condition is thus related to the polynomial growth of the cohomology of the classifying space and loop space of the group. Note that there are finite CW-complexes for which the cohomology of its loop space grows exponentially. This may or may not be an argument against seeing the polynomial growth condition as a natural condition.

2.2   The real χ_c of algebraic stacks

Note that so far we have been defining χ_c for an algebraic stack in terms of χ_c of algebraic spaces. Using the theory of cohomology of compact support for algebraic **k**-stacks of finite type introduced in [OL08b] we shall now see that χ_c may indeed be interpreted as an actual (convergent) Euler characteristic of the stack. We start by checking that some expected properties are indeed true. Following [OL08b] we shall only consider ℓ-adic cohomology (although it seems reasonably clear that their arguments can be carried through also in the case of mixed Hodge structures). We first treat the case of an algebraic stack of finite type (we shall use $H_c^i(\mathcal{X})$ to denote the cohomology of the extension of $\mathcal{X}$ to a separable closure of **k** with $\mathbf{Q}_\ell$-coefficients).

Proposition 2.4 *Let $\mathcal{X}$ be an algebraic stack of finite type.*
  i) If $\mathcal{U} \subseteq \mathcal{X}$ is an open substack and $\mathcal{Z}$ its complement, then we have a long exact sequence

$$\cdots \to H_c^i(\mathcal{U}) \to H_c^i(\mathcal{X}) \to H_c^i(\mathcal{Z}) \to H_c^{i+1}(\mathcal{U}) \to \ldots$$

  ii) If $\dim \mathcal{X} \leq N$, then $H_c^i(X) = 0$ for $i > 2N$.
  iii) $H_c^i(\mathcal{X})$ is mixed of weight $\leq i$.
  iv) $\dim H_c^i(\mathcal{X})$ has polynomial growth.

PROOF: i) follows from [OL08b, 4.9.0.1] and a passage to the limit. As for ii) we can use i) to reduce to the case when $\mathcal{X}$ is smooth and then the result follows from duality. Further, to prove iii) we can use i), and the fact that mixed sheaves are closed under subobjects, quotients and extensions, to reduce to the case when $\mathcal{X}$ is a global quotient. In that case we can find a vector bundle over $\mathcal{X}$ with an open substack which is spatial and with complement of arbitrarily high codimension. As the cohomology of a vector bundle is the cohomology of the base shifted the same amount in degree and weight we can use ii) and i) to reduce to the case an algebraic space in which case it is Deligne's fundamental result, [De80].

Finally, to prove iv) we are reduced to the case when $\mathcal{X}$ is smooth and a global quotient $[X/\mathrm{GL}_n]$. We are then by duality reduced to proving that $\dim H^i(\mathcal{X})$ has polynomial growth which follows from the spectral sequence

$$H^i(\mathrm{BGL}_n) \bigotimes H^j(X) \implies H^{i+j}(\mathcal{X}).$$

$\qquad \square$

Now, [OL08a] defines cohomology with compact support for finite coefficients for algebraic stacks *locally* of finite type. It is easy to see that this cohomology (for constant coefficients say) is the inductive limit of the cohomology over all open substacks of finite type. This property is however destroyed in general when one passes to ℓ-adic or $\mathbf{Q}_\ell$-coefficients. It therefore seems better to *define* $H_c^i(\mathcal{X})$ for a stack only locally of finite type as the inductive limit over open

substacks of finite type. The only thing to decide is in which sense the limit is to be taken, we choose to simply to take the ind-system defined by the open substacks of finite type. This makes sense both in the Galois and the mixed Hodge structure case. (This whole discussion is somewhat moot as in the cases that we shall consider each $H_c^i(\mathcal{X})$ will be an eventually constant system, where an ind-system $\{\mathcal{H}_\alpha\}$ is *eventually constant* if there is an index α so that for any morphism $\alpha \to \beta$, the induced map $\mathcal{H}_\alpha \to \mathcal{H}_\beta$ is an isomorphism. This is turn means that the system is isomorphic as an ind-object to a constant system.)

Theorem 2.5 *i) If $\mathcal{X}$ is essentially of finite type, then for each i, $H_c^i(\mathcal{X})$ is eventually constant.*
ii) If $\mathcal{X}$ is of uniform polynomial growth and $\dim H_c^i(\mathcal{X})$ grows polynomially, then

$$\chi_c(\{\mathcal{X}\}) = \sum_i (-1)^i \{H_c^i(\mathcal{X})\} \in \overline{K}_0^{\mathrm{pol}}(\mathrm{Coh}_{\mathbf{k}}).$$

PROOF: Let $\{\mathcal{X}_i\}$ be a stratification of $\mathcal{X}$ and $\{\mathcal{U}_\alpha\}$ the family of open substacks of finite type. Given N there is only a finite number of $\mathcal{X}_i$ of dimension $> N$. Each of them can be covered by a finite number of the $\mathcal{U}_\alpha$ and as $\{\mathcal{U}_\alpha\}$ is inductive there is a $\mathcal{U}_\alpha$ whose complement has dimension $\le N$ and hence by Proposition 2.4 for any $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$, $H_c^i(\mathcal{U}_\alpha) \to H_c^i(\mathcal{U}_\beta)$ is an isomorphism which proves i).

Continuing, the assumption on polynomial growth and the fact that $H_c^i(\mathcal{X})$ is of weight $\le i$ implies that $\sum_i (-1)^i \{H_c^i(\mathcal{X})\}$ converges and to show that it is equal to $\chi_c(\{\mathcal{X}\})$ it is enough to show this in $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$. This amounts to showing that the weight n-parts are equal. Now, using the notation of i), the weight n part of $\sum_i (-1)^i \{H_c^i(\mathcal{X})\}$ is equal to the weight n-part of $\sum_i (-1)^i \{H_c^i(\mathcal{U}_\alpha)\}$ with $N << 0$. On the other hand, the weight n-part of $\chi_c(\{\mathcal{X}\})$ is equal to the weight n part of

$$\chi_c(\sum_i \{\mathcal{X}_i \cap \mathcal{U}_\alpha\}) = \chi_c(\mathcal{U}_\alpha)$$

as all but a finite number of the $\mathcal{X}_i \cap \mathcal{U}_\alpha$ are empty. Hence we are reduced to the case when $\mathcal{X}$ is of finite type. Cutting $\mathcal{X}$ into pieces we may assume that $\mathcal{X}$ is a global quotient. This means that there are vector bundles over $\mathcal{X}$ with non-spatial locus of arbitrarily high codimension. This high codimension locus may be ignored by (2.1:ii) and (2.4:ii). Hence we are reduced to the case when $\mathcal{X}$ is a space in which case it clear (by definition).   □

3   The motivic class and the refined Euler characteristic

When the base field has characteristic zero we can use the presentation of $K_0(\mathrm{Spc}_{\mathbf{k}})$ (and consequently of $K_0(\mathrm{Stck}_{\mathbf{k}})$) obtained by Bittner (cf., [Bi04]) to get more refined cohomological invariants of classes in $K_0(\mathrm{Spc}_{\mathbf{k}})$. We start by defining the recipients for these invariants. As we are going to use the Bittner representation *in this section* $\mathbf{k}$ *will be a field of characteristic zero*. For a general field $\mathbf{k}$ (of characteristic zero) we let $L_0(\mathrm{Coh}_{\mathbf{k}})$ be the group generated by isomorphism classes in the category of direct limits of bounded constructible $\mathbf{Z}_p$-complexes (in the derived category) over finite type "thickenings" of $\mathbf{k}$ (where p is a fixed prime). We introduce the relations $\{A \bigoplus B\} = \{A\} + \{B\}$ but *no* relations for distinguished triangles. In case $\mathbf{k} = \mathbf{C}$ we alternatively let $L_0(\mathrm{Coh}_{\mathbf{k}})$ be generated by isomorphism classes of bounded $\mathbf{Z}$-complexes A with finitely generated cohomology together with a (polarisable) Hodge structure of weight k on $H^k(A)$ and the same relations as before (and when $\mathbf{k} = \mathbf{R}$ the same plus involutions as above). We define a ring structure on $L_0(\mathrm{Coh}_{\mathbf{k}})$ by putting $\{A\}\{B\} = \{A \bigotimes_{\mathbf{Z}_p}^L B\}$ (resp., of course, $\{A \bigotimes_{\mathbf{Z}}^L B\}$). There is an increasing *degree filtration* on $L_0(\mathrm{Coh}_{\mathbf{k}})$ with $\mathrm{Fil}^k L_0(\mathrm{Coh}_{\mathbf{k}})$ being generated by complexes concentrated in degrees $\le k$ and we let $\widehat{L}_0(\mathrm{Coh}_{\mathbf{k}})$ be the completion of $L_0(\mathrm{Coh}_{\mathbf{k}})$ with respect to this filtration. As the filtration is multiplicative (i.e., $\mathrm{Fil}^k \cdot \mathrm{Fil}^\ell \subseteq \mathrm{Fil}^{k+\ell}$) we have a ring homomorphism $L_0(\mathrm{Coh}_{\mathbf{k}}) \to \widehat{L}_0(\mathrm{Coh}_{\mathbf{k}})$. Before our first major result we need the following lemma.

Lemma 3.1 $\text{Fil}^n K_0(\text{Spc}_\mathbf{k})$ is spanned by classes of smooth and proper varieties of dimension $\leq n$.

PROOF: We prove this by induction over n with $n = -1$ being obvious. Assume now that $\dim U \leq n$, we then want to express it as a linear combination of smooth and proper varieties of dimension $\leq n$. Choosing first a compactification of U and then a resolution of singularities expresses the class as a linear combination of an n-dimensional smooth and proper variety and varieties of smaller dimension which finishes the proof by induction. □

Proposition 3.2 *i) There is a ring homomorphism $\chi_G \colon K_0(\text{Spc}_\mathbf{k}) \to L_0(\text{Coh}_\mathbf{k})$ characterised by $\chi_G(\{X\}) = \{R\Gamma(X)\}$ where X is smooth and proper.*

ii) χ_G extends (uniquely) to a continuous ring homomorphism $\chi_G \colon \widehat{K}_0(\text{Spc}_\mathbf{k}) \to \widehat{L}_0(\text{Coh}_\mathbf{k})$. In particular it extends (uniquely) to a ring homomorphism $\chi_G \colon K_0(\text{Stck}_\mathbf{k}) \to \widehat{L}_0(\text{Coh}_\mathbf{k})$.

iii) For each k there is a group homomorphism $H^k(-) \bigotimes \mathbf{Q} \colon \widehat{L}_0(\text{Coh}_\mathbf{k}) \to \widehat{K}_0(\text{Coh}_\mathbf{k})$ characterised by continuity and the property $H^k(\{C\}) \bigotimes \mathbf{Q} = \{H^k(C) \bigotimes \mathbf{Q}\}$. We have for $x \in \widehat{K}_0(\text{Spc}_\mathbf{k})$ that $H^k(x) \bigotimes \mathbf{Q}$ is pure of weight k and $\chi(x) = \sum_i (-1)^k H^k(\chi_G(x))$. In particular $H^k(\chi_G(x)) \bigotimes \mathbf{Q}$ is the weight k part of $(-1)^k \chi(x)$.

PROOF: By [Bi04, thm. 3.1] $K_0(\text{Spc}_\mathbf{k})$ is generated by classes $\{X\}$, X smooth and proper, and their relations are generated by $\{\text{Bl}_Y X\} - \{E\} = \{X\} - \{Y\}$, where X is smooth and proper, $Y \subseteq X$ a smooth and proper subvariety, $\text{Bl}_Y X$ is the blowing up of X along Y. Now, there is the well-known isomorphism of complexes $R\Gamma(\text{Bl}_Y X) \bigoplus R\Gamma(Y) \cong R\Gamma(X) \bigoplus R\Gamma(Y)$ (e.g., $R\Gamma(-)$ factors through the category of integral motives and the corresponding isomorphism of motives is [Ma68, Cor. §7, Cor. §9]). This proves i).

For ii) we first note that χ_G takes $\mathbb{L}$ to an invertible element in $\widehat{L}_0(\text{Coh}_\mathbf{k})$. Indeed, $\mathbb{L} = \{\mathbf{P}^1\} - 1$ and $\chi_G(\{\mathbf{P}^1\}) = 1 + \{\mathbf{Z}(-1)[-2]\}$ (resp. $\mathbf{Z}_p(-1)[-2]$) and $\{\mathbf{Z}(-1)[-2]\}$ is invertible, with inverse $\{\mathbf{Z}(1)[2]\}$ already in $L_0(\text{Coh}_\mathbf{k})$. This gives us a ring homomorphism $K_0(\text{Spc}_\mathbf{k})[\mathbb{L}^{-1}] \to \widehat{L}_0(\text{Coh}_\mathbf{k})$ and we finish if we can show that it is continuous in the filtration topologies. This however follows directly from Lemma 3.1 and the facts that $\chi_G(\{X\}/\mathbb{L}^m) = \{R\Gamma(X)(m)[2m]\}$ and that $R\Gamma(X)$ is of amplitude $[0, 2n]$ if X is n-dimensional. The only thing left is then the uniqueness of the extension of χ_G to $K_0(\text{Stck}_\mathbf{k})$ but it follows from the fact that $K_0(\text{Spc}_\mathbf{k}) \to K_0(\text{Stck}_\mathbf{k})$ is a localisation.

For the last part, as $H^k(-)$ is additive on complexes we get an induced map $H^k(-) \bigotimes \mathbf{Q} \colon L_0(\text{Coh}_\mathbf{k}) \to K_0(\text{Coh}_\mathbf{k})$ which clearly is continuous and hence extends to $\widehat{L}_0(\text{Coh}_\mathbf{k})$. Now both $H^k(\chi_G(-))$ is additive and continuous so to verify that it maps to the weight k part of $\widehat{L}_0(\text{Coh}_\mathbf{k})$ it is enough to very it on classes $H^k(\chi_G(\{X\})) \bigotimes \mathbf{Q}$ where X is smooth and proper. In that case $H^k(\chi_G(\{X\})) \bigotimes \mathbf{Q} = \{H^k(X, \mathbf{Q})\}$ (resp. with $\mathbf{Q}_p$-coefficients) and $H^k(X, \mathbf{Q})$ is indeed pure of weight k. This implies that the sum $\sum_i (-1)^k H^k(\chi_G(x))$ converges and gives a continuous additive function on $\widehat{K}_0(\text{Spc}_\mathbf{k})$ and so does χ. Hence to verify that they are equal it suffices to do it on $\{X\}$, X smooth and proper, in which case it is clear. □

Remark: Instead of the Bittner presentation one could use earlier results (cf., [GS96]) that associates a virtual motive to a class of $K_0(\text{Coh}_\mathbf{k})$.

In order to detect non-trivial elements of $L_0(\text{Coh}_\mathbf{k})$ (or $\widehat{L}_0(\text{Coh}_\mathbf{k})$) it is useful to pass to the underlying abelian groups. Hence, we let $L_0(\text{Cplx})$ be the group generated by isomorphism classes of bounded complexes (in the derived category) of finitely generated $\mathbf{Z}$-complexes (resp., of course, $\mathbf{Z}_p$-complexes) by the same relations as before and $\widehat{L}_0(\text{Cplx})$ the corresponding degree completion. Again they have a ring structure given by the derived tensor product. We also let $L_0(\text{Ab})$ be the corresponding group based on modules rather than complexes but we do not give it a multiplicative structure. We have a group homomorphism $L_0(\text{Ab}) \to L_0(\text{Cplx})$ given by $\{M\} \mapsto \{M[0]\}$ and $H^k \colon \widehat{L}_0(\text{Cplx}) \to L_0(\text{Ab})$ given by $\{C\} \mapsto \{H^k(C)\}$. We continue by defining $1 := \{\mathbf{Z}\} \in L_0(\text{Ab})$ (which maps to the identity element of $L_0(\text{Cplx})$) and $\alpha_{p,n} := \{\mathbf{Z}/q^n\}$, where $n > 0$ and q is a prime. Finally, for M an abelian group we let $M[t, t^{-1}]$ be

the group of finite formal sums $\sum_{i \in \mathbf{Z}} m_i t^i$, $m_i \in M$ and $M((t^{-1}))$ the group of formal sums $\sum_{i \in \mathbf{Z}} m_i t^i$ which possibly have an infinite number of negative power terms.

Proposition 3.3 *i)* $L_0(\mathrm{Ab})$ *is the free abelian group on* 1 *and the* $\alpha_{q,n}$, *where* q *runs over all primes (resp. is equal to* p).

ii) The maps $H^\bullet \colon L_0(\mathrm{Cplx}) \to L_0(\mathrm{Ab})[t, t^{-1}]$ *and* $H^\bullet \colon \widehat{L}_0(\mathrm{Cplx}) \to L_0(\mathrm{Ab})((t^{-1}))$ *given by* $H^\bullet(\{C\}) = \sum_i H^i(\{C\}) t^i$ *are isomorphisms.*

iii) The multiplicative structure on $L_0(\mathrm{Ab})[t, t^{-1}]$ *and* $L_0(\mathrm{Ab})((t))$ *induced by these isomorphisms is characterised by the conditions that the maps* $\mathbf{Z}[t, t^{-1}] \to L_0(\mathrm{Ab}_{t, t^{-1}})$ *and* $\mathbf{Z}((t^{-1})) \to L_0(\mathrm{Ab})((t))$ *induced by* $m \mapsto m \cdot 1$, $t \mapsto t$ *and the second map is continuous are ring homomorphisms and*

$$\alpha_{q,m} \alpha_{q',n} = \begin{cases} 0 & \text{if } q \neq q' \\ \alpha_{q,\min(m,n)}(1 + t^{-1}) & \text{if } q = q' \end{cases}$$

together with the fact that the multiplication is continuous in the t^{-1}*-adic topology in the* $((t^{-1}))$*-case.*

PROOF: As the rings $\mathbf{Z}$ and $\mathbf{Z}_p$ have global dimension 1 the complexes in question are isomorphic to the direct sum of their (shifted) cohomology modules. This gives the second part. The first part follows from the classification of finitely generated modules over the rings in question and the last from the basic calculation of tensor products and Tor-groups. □

Remark: i) The conclusion should be of course thought of as saying that the usual rank and torsion invariants of the cohomology of a smooth and proper variety are in fact invariants of its class in $K_0(\mathrm{Spc}_{\mathbf{k}})$ (and in fact of its class in $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$). The precise structure of the groups are being mentioned mainly to emphasise how very much larger they are than the ordinary Grothendieck groups (which are just isomorphic to $\mathbf{Z}$).

ii) If one works rationally instead of integrally, then the refined Euler characteristic can be recovered from the ordinary Euler class in $K_0(\mathrm{Coh}_{\mathbf{k}})$. Indeed, $R\Gamma(X)$ is a pure complex so is isomorphic to the sum of its (shifted) cohomology and then we can use Theorem 3.2. Hence, the extra information provided by the refined Euler characteristic is torsion and integrality information.

There are a number of variants of this technique some of which will appear in the next result. To prepare for it let us note that there is an obvious additive Bittner type presentation of $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$: It is generated by classes $\{X\}/\mathbb{L}^n$, $n \geq 0$, and relations $\{\mathrm{Bl}_Y X\}/\mathbb{L}^n - \{X\}/\mathbb{L}^n = \{E\}/\mathbb{L}^n - \{Y\}/\mathbb{L}^n$ as before and $\{\mathbf{P}^1 \times X\}/\mathbb{L}^{n+1} - \{X\}/\mathbb{L}^{n+1} = \{X\}/\mathbb{L}^n$ for smooth and proper $Y \subseteq X$. Also we shall put $\widehat{\mathbf{Z}} := \varprojlim_n \mathbf{Z}/n = \prod_p \mathbf{Z}_p$ and $H^*(X, \widehat{\mathbf{Z}}) := \prod_p H^*(X, \mathbf{Z}_p)$.

Theorem 3.4 *i) Let* $A_{\mathbf{k}}$ *be the group generated by isomorphism classes of algebraic group schemes* A *over* $\mathbf{k}$ *whose connected component is an abelian variety and whose group of geometric connected components has a finitely generated group of geometric points and with relations* $\{A \bigoplus B\} = \{A\} + \{B\}$. *Then there is a (unique) group homomorphism* $\mathrm{Pic}_{\mathbf{k}} \colon K_0(\mathrm{Spc}_{\mathbf{k}}) \to A_{\mathbf{k}}$ *for which* $\mathrm{Pic}_{\mathbf{k}}(\{X\}) = \{\mathrm{Pic}(X)\}$ *for a smooth and proper* X.

ii) There is an extension of $\mathrm{Pic}_{\mathbf{C}}$ *to* $\widehat{K}_0(\mathrm{Spc}_{\mathbf{C}})$ *vanishing on* $\mathrm{Fil}^0 \widehat{K}_0(\mathrm{Spc}_{\mathbf{C}})$ *(which we shall also call* $\mathrm{Pic}_{\mathbf{C}}$).

iii) Let $L_0(\mathrm{Ab}_{\mathbf{k}})$ *be the group generated by isomorphism classes of étale algebraic group schemes* A *whose group of geometric connected components has a finitely generated group of geometric points and with relations* $\{A \bigoplus B\} = \{A\} + \{B\}$. *Then there are continuous group homomorphisms* $\mathrm{NS}^k \colon \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}}) \to L_0(\mathrm{Ab}_{\mathbf{k}})$ *characterised by* $\mathrm{NS}^k(\{X\}/\mathbb{L}^n) = \mathrm{NS}^{k+n}(\{X\})$ *and* $\mathrm{NS}^k(\{X\}) = \{\mathrm{NS}^k(X)\}$, *for* X *smooth and proper, where* $\mathrm{NS}^k(X)(\bar{\mathbf{k}})$ *is the group of algebraic codimension* k*-cycles on* $X_{\bar{\mathbf{k}}}$ *modulo homological equivalence with its natural* $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$*-action.*

PROOF: Starting with i) it follows directly from the Bittner presentation and the computation of the Picard scheme of a blow up. As for ii) we define $\mathrm{Pic}'(\{X\}/\mathbb{L}^n)$, X smooth and proper, as follows: We let it be the class of $A_X^{0,n} \bigoplus A_X^{c,n}$, where $A_X^{c,n}$ is the inverse image of the classes of

type $(n+1, n+1)$ in $H^{2n+2}(X, \mathbf{C})$ under the map $H^{2n+2}(X, \mathbf{Z}) \to H^{2n+2}(X, \mathbf{C})$. Similarly, we let $A_X^{0,n}$ be the Weil intermediate Jacobian (cf., [We52]) associated to the (polarisable) Hodge structure on $H^{2n+1}(X, \mathbf{Z})$. Now $\mathrm{Pic}(X)$, as any complex commutative algebraic group whose group of components is an abelian variety, is the direct product of its connected component and its group of components (as we are working over an algebraically closed field). Furthermore, $A_X^{0,1}$ is the Jacobian, i.e., the connected component of $\mathrm{Pic}(X)$, and $A^{c,1}(X)$ is the group of components of $\mathrm{Pic}(X)$ by Lefschetz theorem. It is furthermore clear that $\{\mathbf{P}^1 \times X\}/\mathbb{L}^{n+1} - \{X\}/\mathbb{L}^{n+1}$ is mapped to the same element as $\{X\}/\mathbb{L}^n$ which shows that we Pic' is given as a map from $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$. Finally, as $R\Gamma(X)$ is of amplitude $[0, 2\dim X]$ it is clear (using Lemma 3.1)that $\mathrm{Fil}^0 K_0(\mathrm{Spc}_{\mathbf{k}})$ is mapped to zero which in particular gives us an extension to $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ with the same property.

Turning to iii), by definition $\mathrm{NS}^k(X)$, for X smooth and proper, is the subgroup of $H^{2k}(X, \widehat{\mathbf{Z}}(k))$ spanned by the cycle classes of k-codimensional subvarieties of $X_{\bar{\mathbf{k}}}$. It is clearly stable by the Galois action and hence gives rise to an étale group scheme. The next step is to show that it is finitely generated. If not there is a countable set $\{Y_i\}$ of k-codimensional subvarieties whose cycle classes span a subgroup M of $H^{2k}(X, \widehat{\mathbf{Z}}(k))$ with the property that there is no finite subset of Y_i whose cycle classes span M. The subvarieties Y_i may be defined over a common countably generated algebraically closed subfield K of $\bar{\mathbf{k}}$ which then may be embedded in $\mathbf{C}$. We may then replace $\bar{\mathbf{k}}$ by $\mathbf{C}$ in which case the cycle class map factors through $H^{2k}(X, \mathbf{Z}) \to H^{2k}(X, \widehat{\mathbf{Z}}(k))$ which shows that M is finitely generated which is a contradiction.

What remains to be verified are the Bittner relations and $\{\mathbf{P}^1 \times X\}/\mathbb{L}^{n+1} - \{X\}/\mathbb{L}^{n+1} = \{X\}/\mathbb{L}^n$. These follow immediately from the formulas for the cohomology and Chow groups of blowing ups and products with $\mathbf{P}^1$ and the compatibility of these formulas with the cycle map. $\square$

Remark: i) Instead of the Weil intermediate Jacobian we could have used the maximal abelian subvariety of the usual (="Griffiths") intermediate Jacobian $H^{2n+1}(X, \mathbf{C})/H^{2n+1}(X, \mathbf{Z}) + F^n H^{2n+1}(X, \mathbf{C})$.

ii) The maps NS^k will be used in [Ek09b] to fit a counterexample of Swan to Noether's problem into our context.

Corollary 3.5 *The rings $K_0(\mathrm{Spc}_{\mathbf{k}})$, $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$, $K_0(\mathrm{Stck}_{\mathbf{k}})$ and $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ contain zero-divisors.*

PROOF: Let us recall the setup of [Po02] (where Poonen proves the statement for $K_0(\mathrm{Spc}_{\mathbf{k}})$). We have an abelian variety A over $\mathbf{k}$ with endomorphism ring (over $\mathbf{k}$ as well as geometrically) equal to the ring of integers R in a number field. Furthermore, we have a non-principal ideal I in R and put $B := I \bigotimes_R A$ (which in particular has the property that $\mathrm{Hom}(A, B) = I$ as R-module). We have that $B^n \cong I^{\oplus n} \bigotimes_R A \cong A^{n-1} \bigoplus A \bigotimes_R I^n$ and if we choose n such that I^n is principal ($n = 2$ in Poonen's specific example) we get $B^n \cong A^n$ which gives $0 = (\{A\}^{n-1} + \{A\}^{n-2}\{B\} \cdots + B^{n-1})(\{A\} - \{B\})$ in $K_0(\mathrm{Spc}_{\mathbf{k}})$ and we want to show that both factors have non-zero images in any of the rings in question and as all of them map compatibly to $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ it is enough to show that their images in that ring are non-zero. For the first factor this is trivial, the weight $2\dim A(n-1)$-part of its Euler characteristic is non-trivial. We are thus left with the problem on whether $\{A\} = \{B\} \in \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$. Now, to show that they are different it is enough to show that they are distinct modulo Fil^0 and hence as elements of $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$. For this we may by standard limit arguments assume that $\mathbf{k} = \mathbf{C}$. As $\mathrm{Pic}(\{A\}) = \{A\} \in A_{\mathbf{C}}$ and the same for B we are reduced to showing that $A \bigoplus C \cong B \bigoplus C$ for some abelian group scheme C whose connected component is an abelian variety. Such an isomorphism induces an isomorphism $\mathrm{Hom}(A, A) \bigoplus \mathrm{Hom}(A, C) \cong \mathrm{Hom}(A, B) \bigoplus \mathrm{Hom}(A, C)$ of finitely generated R-modules. Putting $M := \mathrm{Hom}(A, C)$ this gives an isomorphism $R \bigoplus M \cong I \bigoplus M$ but we have cancellation for finitely generated R-modules so that we get an isomorphism $R \cong I$ which contradicts the assumption that I is non-principal. $\square$

It is natural to ask in what generality the last argument works. The general question is to what extent a group scheme is determined by its class in $A_{\mathbf{k}}$. This is the same problem as asking

whether $A \bigoplus C \cong B \bigoplus C$ implies that $A \cong B$. We restrict ourselves to the case when A and B are abelian varieties and by restricting to identity components we may assume that C is also an abelian variety.

Proposition 3.6 *Assume that A, B and C are abelian varieties and $A \bigoplus C \cong B \bigoplus C$. Then there is a locally free rank 1 module I over $R := \operatorname{End}(A)$ such that B is isomorphic to $I \bigotimes_R A$ so that in particular $A \cong B$ precisely when $R \cong I$ and a torsion free finitely generated R-module M such that $R \bigoplus M \cong I \bigoplus M$. In particular if R is hereditary and no simple factor of $R \bigotimes \mathbf{Q}$ is a totally definite quaternion algebra over its centre then $A \cong B$.*

PROOF: We may assume that $\mathbf{k}$ is finitely generated. We get for any prime p an isomorphism $T_p A \bigoplus T_p C \cong T_p B \bigoplus T_p C$ of modules over the Galois group $\mathcal{G}$ of $\mathbf{k}$. The Krull-Schmidt theorem then implies that we have an isomorphism $T_p A \cong T_p B$ of $\mathcal{G}$-modules. Let $I := \operatorname{Hom}(A, B)$. By the Tate conjecture [Fa83, Thm 4] we have $I_p := I \bigotimes \mathbf{Z}_p = \operatorname{Hom}_{\mathcal{G}}(T_p A, T_p B)$ and in particular, using the isomorphism $T_p A \cong T_p B$ we get that $I_p \cong R_p := R \bigotimes \mathbf{Z}_p$ as R_p-modules so that I is a locally free R-module of rank 1. This implies first that the evaluation map $\operatorname{Hom}_{\mathcal{G}}(T_p A, T_p B) \bigotimes_{R_p} T_p A \to T_p B$ and second that $T_p(I \bigotimes_R A) = I \bigotimes_R T_p(A)$. Together these facts imply that the tautological map $I \bigotimes_R A \to B$ induces an isomorphism when $T_p(-)$ for all p is applied and hence is an isomorphism which gives the first part. Putting $M := \operatorname{Hom}(A, C)$ we get an isomorphism of R-modules $R \bigoplus M = \operatorname{Hom}(A, A \bigoplus C) \cong \operatorname{Hom}(A, B \bigoplus C) = I \bigoplus M$.

Finally, when R is hereditary the fact that M is torsion free implies that M is projective. The condition on $R \bigotimes \mathbf{Q}$ implies that it fulfils the Eichler condition relative to $\mathbf{Z}$ (see [Re75, Def. 38.1]) and hence we have cancellation for projective modules [loc. cit., Thm. 38.2].   □

Remark: Cancellation is not always true, see [Sh77]. Its characteristic zero example involves a non-hereditary order. However, the purely algebraic consequences of one of its positive characteristic example implies that there is an example where R is a maximal order in a definite quaternion algebra over $\mathbf{Q}$.

Recall that the Kummer sequence (resp. the exponential sequence) gives a group homomorphism from the Brauer group of a variety X (resp. over $\mathbf{C}$) to $H^3(X, \widehat{\mathbf{Z}})$ (resp. $H^3(X, \mathbf{Z})$). The image of an element of the Brauer group under this map will be called its *characteristic class*.

Proposition 3.7   *Suppose $\mathbf{k}$ is algebraically closed. For every integer $n \geq 1$ for which $n + 1$ is not a power of the characteristic of $\mathbf{k}$ there is a smooth and projective variety X and a projective space fibration $P \to X$ of relative dimension n and with non-trivial characteristic class.*

PROOF: Suppose that we have a finite group G of order prime to the characteristic if the latter is positive acting freely on a variety Y and a projective representation $\pi \colon G \to \operatorname{PGL}_{n+1}(\mathbf{k})$. We then get a projective space bundle $P := Y \times_G \mathbf{P}^n \to Y/G =: X$ and it is clear that its characteristic class is the inverse image under the classifying map $X \to BG$ of $Y \to X$ of the characteristic class of π. The latter is defined as the image of π under the composite $H^1(G, \operatorname{PGL}_{n+1}(\mathbf{k})) \to H^2(G, \mathbf{k}^*) \to H^2(G, \bar{\mathbf{k}}^*) = H^2(G, \mathbf{Q}/\mathbf{Z}) = H^3(G, \mathbf{Z})$.

Now I claim that for every positive integer K there is a smooth and projective Y as above such that the induced map $H^i(G, \mathbf{Z}) \to H^i(X, \widehat{\mathbf{Z}})$ is injective for $i \leq K$. Indeed, the Godeaux-type construction of Serre (cf., [Se58, §20]) starts with a linear representation V of G whose associated projective representation is faithful and constructs a smooth complete intersection Y in the projective space $\mathbf{P}(V)$ of V on which G acts freely. The maximal possible dimension of Y depends on the codimension of the points of $\mathbf{P}(V)$ with non-trivial stabiliser. That codimension may be made as large as possible by choosing a suitable V (for instance by replacing a given V by a high power of it) and therefore we may find Y's of arbitrarily high dimension. Consider first the stack quotient $[\mathbf{P}(V)/G]$. The natural map $[\mathbf{P}(V)/G] \to BG$ makes $[\mathbf{P}(V)/G]$ a projective bundle over BG associated to the vector bundle $[V/G] \to BG$. This implies that if $\xi \in H^2([\mathbf{P}(V)/G], \widehat{\mathbf{Z}})$ is the first Chern class of the G-linearised line bundle $\mathcal{O}(1)$, then $H^*([\mathbf{P}(V)/G], \widehat{\ })$ is the free $H^*(G, \widehat{\mathbf{Z}})$-module on the ξ^i, $i = 0, \ldots, n-1$ (cf., [Gr68, 1.9]). (There may be some doubt about the $\mathbf{Z}_p$-part of this when $p = \operatorname{char} \mathbf{k}$. It turns out to be true but is not used in the subsequent arguments.)

We now have a morphism $X = Y/G = [Y/G] \to [\mathbf{P}^n/G]$ induced by the inclusion $Y \subseteq \mathbf{P}^n$ giving rise to a comparison map from the spectral sequence $H^i(G, H^j(\mathbf{P}^n, \widehat{\mathbf{Z}})) \implies H^{i+j}([\mathbf{P}^n/G], \widehat{\mathbf{Z}})$ to the corresponding one for $[Y/G]$. By the Lefschetz theorem the map $H^i(\mathbf{P}^n, \widehat{\mathbf{Z}}) \to H^i(Y, \widehat{\mathbf{Z}})$ is an isomorphism for $i < \dim Y$ which by the comparison map of spectral sequences implies that $H^i([\mathbf{P}^n/G], \widehat{\mathbf{Z}}) \to H^i(X, \widehat{\mathbf{Z}})$ is an isomorphism for $i < \dim Y$. In particular, the composite $H^i(G, \widehat{\mathbf{Z}}) \to H^i(X, \widehat{\mathbf{Z}})$ is injective in the same range. As we can make $\dim Y$ arbitrarily large we have proven our claim.

Let now $1 \to \mathbf{k}^* \to H \to G \to 1$ be a non-trivial central extension of G and assume that we have a linear representation $H \to \mathrm{GL}_{n+1}(\mathbf{k})$ of H which takes $\mathbf{k}^*$ to scalar matrices. This gives us an induced projective representation $G \to \mathrm{PGL}_{n+1}$. I claim that the characteristic class of this representation is equal to the class of the central extension H. Indeed, while the most conceptual would no doubt be "gerbic", the simplest is probably the easy computation by cocycles. In particular it is non-trivial as the central extension is. Now, choose a Y as above with $K \geq 3$. The pullback of α to X represents the projective bundle $Y \times_G \mathbf{P}^n \to Y/G$ and by assumption it is non-zero. If $n+1$ is not divisible by the characteristic of $\mathbf{k}$ we can let H be the $g = 1$ level $n+1$ theta group so that $G = \mathbf{Z}/(n+1)\mathbf{Z} \times \mathbf{Z}/(n+1)\mathbf{Z}$ and the theta group has an $n+1$-dimensional representation of the required kind. If $n+1$ is not a power of the characteristic we may let d be any non-trivial divisor of $n+1$ not divisible by the characteristic and let H be the level d theta group and let the $n+1$-dimensional representation be $(n+1)/d$ copies of the standard representation. $\square$

Remark: i) As we only need $K \geq 3$ (with the notations of the proof) the proof actually gives us 3-dimensional examples.

ii) There are 2-dimensional examples at least for some values of n. One such example is an Enriques surface. In that case Poincaré duality shows that the torsion subgroup of $H^3(X, \widehat{\mathbf{Z}})$ is equal to $\mathbf{Z}/2$ and it comes from the Brauer group (as the cohomological Brauer group is equal to the Brauer group). I don't know however which n can be chosen in this case (presumably $n = 1$ is possible).

Lemma 3.8 *Let G be a semi-simple group over an algebraically closed field $\mathbf{k}$.*

i) $\{G\}$ is invertible in $K_0(\mathrm{Spc}_{\mathbf{k}})'$.

ii) Assume that F is a G-space and $P \to X$ is a G-torsor (of spaces). If we for the two G-torsors $P \to X$ and $P \times F \to P \times_G F$ have the equalities $\{P\} = \{G\}\{X\}$ and $\{P \times F\} = \{G\}\{P \times_G F\}$, then we have $\{P \times_G F\} = \{F\}\{X\}$.

PROOF: If we let B be a Borel subgroup of G we have a B-torsor $G \to G/B$ and as B is a successive extension of $\mathbf{G}_m$'s and $\mathbf{G}_a$'s and thus $B \in \mathcal{Z}\mathrm{ar}$ so that $\{G\} = \{B\}\{G/B\}$. Now, again by the fact B is a successive extension $\mathbf{G}_m$'s and $\mathbf{G}_a$'s we have that $\{B\}$ is an integer polynomial in $\mathbb{L}$ and as G/B has a cell decomposition so does $\{G/B\}$. This polynomial is determined by the fact that $\chi_c(\{G\})$ is the same polynomial in $q := \chi_c(\mathbb{L})$ and it is well-known that this polynomial is a power of q times a product of $q^n - 1$'s which shows that $\{G\}$ is indeed invertible in $K_0(\mathrm{Spc}_{\mathbf{k}})'$.

As for the second part we have, using the assumed relations,

$$\{G\}\{F\}\{X\} = \{F\}\{P\} = \{P \times F\} = \{G\}\{P \times_G F\}$$

and by the first part we can cancel $\{G\}$. $\square$

Proposition 3.9 *Assume that $\mathbf{k}$ is algebraically closed. There are PSL_n-torsors, for any $n \geq 1$ not a power of the characteristic of $\mathbf{k}$, $T \to X$ over smooth and projective varieties X such that $\{T\} \neq \{\mathrm{PSL}_n\}\{X\} \in \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$. In particular the map $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}}) \to \widehat{K}_0^{\mathrm{PSL}_n}(\mathrm{Spc}_{\mathbf{k}})$ is not injective and similarly for its variants.*

PROOF: By Proposition 3.7 there is a $\mathbf{P}^{n-1}$-fibration $\pi \colon P \to X$, X smooth and proper, whose characteristic class is non-trivial. The pullback of $P \to X$ along $P \to X$ has a section and hence has a reduction of the structure group to GL_{n+1} and in particular the class α pulls back to 0 in

$H^3(P, \mathbf{Z}_p)$. Now, we have that the $R^i\pi_*\mathbf{Z}_p$ are constant as they become constant after pullback along π which has connected fibres. As also $R^{2i+1}\pi_*\mathbf{Z}_p = 0$, the Leray spectral sequence for π gives an exact sequence

$$H^3(X, \mathbf{Z}_p) \to H^3(P, \mathbf{Z}_p) \to H^1(X, R^2\pi_*\mathbf{Z}_p) = H^1(X, \mathbf{Z}_p).$$

Now, tensored with $\mathbf{Q}$, the spectral sequence degenerates (for instance for reasons of weight) so that the kernel of $H^3(X, \mathbf{Z}_p) \to H^3(P, \mathbf{Z}_p)$ is a torsion group and furthermore $H^1(X, \mathbf{Z}_p)$ is torsion free. Together this shows that the torsion subgroup of $H^3(X, \mathbf{Z}_p)$ maps surjectively onto that of $H^3(P, \mathbf{Z}_p)$. However, we have seen that α goes to zero so the conclusion is that the torsion subgroup of $H^3(P, \mathbf{Z}_p)$ is strictly smaller than that of $H^3(X, \mathbf{Z}_p)$. This implies that we do *not* have that $\{P\} = \{\mathbf{P}^n\}\{X\} \in \widehat{K}_0(\mathrm{Spc}_\mathbf{k})$ as otherwise $H^\bullet(\chi_G(P)) = (1 - s^{n+1})/(1 - s)H^\bullet(X)$, where $s := t^2$, and in particular $\{H^3(P, \mathbf{Z}_p)\}$ would be equal to $\{H^3(X, \mathbf{Z}_p)\} + \{H^1(X, \mathbf{Z}_p)\}$. This would give that the torsion subgroup of $H^3(P, \mathbf{Z}_p)$ would be equal to that of $H^3(X, \mathbf{Z}_p)$ which we have seen is not the case.

Following Lemma 3.8 one of the two PSL_{n+1} torsors, $T \to X$ associated to $P \to X$ and $T \times \mathbf{P}^n \to T \times_{\mathrm{PSL}_{n+1}} \mathbf{P}^n = P$ do not fulfil the multiplicativity relation. $\quad\square$

Remark: i) Lemma 3.8 is necessary only to pinpoint the torsors for which multiplicativity fails. If one doesn't care about that one could, by way of contradiction, assume multiplicativity for all PSL_{n+1}-torsors which implies that $\widehat{K}_0(\mathrm{Spc}_\mathbf{k}) \to K_0^{\mathrm{PSL}_{n+1}}(\mathrm{Spc}_\mathbf{k})$ is an isomorphism. We can then apply (1.1:vi).

ii) Similar results are true for every semi-simple special algebraic group and will be shown elsewhere.

4   A sample computation

We shall now give an example in the spirit of [Be06] (though the cases covered by Bergström are more complicated than the one we shall discuss). The general technique is the following: Present the stack whose class we want to compute as the stack quotient of a linear group acting (linearly) on an open subset of a linear space (or possibly projective space, though that would typically give us PGL_n as linear group which does have torsors not trivial in the Zariski topology). The class of the stack quotient for the action of the group on the full linear space is equal to a power of $\mathbb{L}$ divided by the class of the group. Hence, we may look instead at the complement of the open subset. That complement can then be stratified (typically by imposing various kinds of singular sets) and for each stratum the data specifying it can be put in standard position (which just means a reduction of the structure group). A stratum with data in fixed position can then be embedded as an open subset of a linear space with group action and one is again reduced to considering the complement. Unfortunately the number of strata that one must consider very quickly becomes large so I shall be somewhat sketchy as my aim is only to illustrate the technique.

Example: We shall need to compute $\{\mathrm{B}\Sigma_2\}$ and $\{\mathrm{B}\Sigma_3\}$. In [Ek09b] we shall prove that in general $\{\mathrm{B}\Sigma_n\} = 1$ for all n but these cases can be done by hand. As we are going to make that restriction anyway we assume that the characteristic is different from 2 and 3. For $\Sigma_2 \cong \{\pm 1\}$ we can embed it in $\mathbf{G}_m \in \mathcal{Z}$ar and hence by Proposition 1.1 we get $\{\mathrm{B}\Sigma_2\} = \{\mathbf{G}_m/\pm 1\}\{\mathrm{B}\mathbf{G}_m\} = \{\mathbf{G}_m\}\{\mathrm{B}\mathbf{G}_m\}$ and $1 = \{\mathbf{G}_m\}\{\mathrm{B}\mathbf{G}_m\}$ which gives $\{\mathrm{B}\Sigma_2\} = 1$. For $\mathrm{B}\Sigma_3$ we have that the standard representation divided by the trivial subrepresentation gives an embedding $\Sigma_3 \subset \mathrm{GL}_2 \in \mathcal{Z}$ar and hence by Proposition 1.1 we get that $\{[\mathbf{P}^1/\Sigma_3]\} = \{\mathbf{P}^1\}\{\mathrm{B}\Sigma_3\}$. Now, there are three points of $\mathbf{P}^1$ with non-trivial Σ_3-fixed points and if U is the complement of them we have $[U/\Sigma_3] = U/\Sigma_3$ and U/Σ_3 is $\mathbf{P}^1$ minus one point. Hence if $X := \mathbf{P}^1 \setminus U$ we have $\{[\mathbf{P}^1/\Sigma_3]\} = \{U/\Sigma_3\} + \{[X/\Sigma_3]\} = \{\mathbf{P}^1\} - 1 + \{[X/\Sigma_3]\}$. As X consists of three points permuted by Σ_3 according to the standard permutation representation and hence $[X/\Sigma_3] \cong \mathrm{B}\Sigma_2$ and by what we just proved $\{[X/\Sigma_3]\} = 1$

and thus $\{[\mathbf{P}^1/\Sigma_3]\} = \{\mathbf{P}^1\}$ and thus we get $\{\mathbf{P}^1\} = \{\mathbf{P}^1\}\{B\Sigma_3\}$ and as $\{\mathbf{P}^1\} = \mathbb{L} - 1$ it is invertible so we get $\{B\Sigma_3\} = 1$.

Let $\mathcal{M}$ be the algebraic stack of smooth and proper genus 1 curves with a polarisation of degree 3. Here *polarised* means a curve together with a line bundle. To the universal pair $(\mathcal{C} \to \mathcal{M}, \mathcal{L})$ we associate the rank 3 vector bundle $\pi_*\mathcal{L}$ (where π is the projection $\mathcal{C} \to \mathcal{M}$). The frame bundle E of $\pi_*\mathcal{L}$ is spatial, in fact the space of non-singular cubic forms in three variables, and we have that $\mathcal{M}$ is the stack quotient $[E/\mathrm{GL}_3]$. Now, let V be the space of cubic forms and V' the closed subspace of singular cubic forms. Thus we have

$$\mathbb{L}^{10}\{\mathrm{GL}_3\}^{-1} = \{[V/\mathrm{GL}_3]\} = \{[V'/\mathrm{GL}_3]\} + \{\mathcal{M}\}$$

and as the left hand side is known, we focus attention on $\{[V'/\mathrm{GL}_3]\}$. To continue we assume, for simplicity, that char $\mathbf{k} \neq 2, 3$. Inside of V' we have the open subset V_1' of cubics with exactly one singular point. In turn V_1' can be divided up into to the nodal part V_n', the open subset where the singular locus is reduced and the cuspidal part V_c', where also the second derivatives of the cubic vanish at the singular point. In both cases one may fix a point, (0:1:0), and let U_n' and U_c' be the set of cubics having the fixed point as unique singular point which is nodal resp. cuspidal. In the nodal case the singular locus gives a section and from that it easily follows that the natural map $U_n' \to [V_n'/\mathrm{GL}_3]$ is smooth and surjective and from that one gets an isomorphism $[U_n'/G] \xrightarrow{\sim} [V_n'/\mathrm{GL}_3]$, where G is the stabiliser in GL_3 of the fixed point. Similarly, in the cuspidal case, the vanishing of the cubic, the first and the second derivatives defines a section giving an isomorphism $[U_c'/G] \xrightarrow{\sim} [V_c'/\mathrm{GL}_3]$. These isomorphisms of stacks of course give equalities $\{[U_n'/G]\} = \{[V_n'/\mathrm{GL}_3]\}$ and $\{[U_c'/G]\} = \{[V_c'/\mathrm{GL}_3]\}$ and we can then make a reassembly: We let U_1' be the space of cubics with a unique singularity at the fixed point and we then get

$$\{[U_1'/G]\} = \{[U_n'/G]\} + \{[U_c'/G]\} = \{[V_n'/\mathrm{GL}_3]\} + \{[V_c'/\mathrm{GL}_3]\} = \{[V_1'/\mathrm{GL}_3]\}.$$

Continuing we let U be the linear space of cubics with a singular point at (0:1:0) so that U_1' is an open subset of U. Again we get that $\{[U/G]\} = \mathbb{L}^7\{G\}^{-1}$, as $G \in \mathcal{Z}\text{ar}$ being the extension of groups that are, (and as scheme $G \cong \mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathbf{A}^2$ so that $\{G\} = \{\mathrm{GL}_2\}\{\mathrm{GL}_1\}\mathbb{L}^2$). We are therefore reduced to considering the complement U'' of U_1' in U. Going back to an earlier reduction we also need to consider the complement V'' of V_1' in V'. This leads to the following cases:

- We have a stratum W in V consisting (geometrically) of three distinct lines not all passing through a point. As $\{[W/\mathrm{GL}_3]\} = \{W\}\{\mathrm{GL}_3\}^{-1}$ it is enough to compute $\{W\}$. Letting $\overline{W}$ be the quotient of W by the action by the scalars, the quotient map $W \to \overline{W}$ is a $\mathbf{G}_m$-torsor so we have $\{W\} = (\mathbb{L} - 1)\{\overline{W}\}$ and it will be enough to compute $\{\overline{W}\}$. We have a map from W', the space of linearly independent triples of vectors (a.k.a. invertible matrices) in $(\mathbf{A}^3)^3$, to W, given by thinking of the vectors as linear forms and multiplying them together. We can act by the semi-direct product $H := \mathbf{G}_m^3 \ltimes \Sigma_3$ on W', multiplying each vector by an invertible scalar and permuting the three vectors. The action is free and we have $\overline{W} = W'/H$. The action is also isomorphic to left multiplication of H on GL_3, where H is identified with the normaliser of the group of diagonal matrices in GL_3. Hence we get from (1.1:ix) that $\{W\} = \{BH\}\{\mathrm{GL}_3\}$ and it remains to compute $\{BH\}$. For that we consider the natural linear action of H on $V = \mathbf{k}^3$. As usual we have that $\{[V/H]\} = \mathbb{L}^3\{BH\}$. The diagonal matrices of H act freely and transitively on the open subset V_0 of V defined by all coordinates being non-zero. This gives that $[V_0/H] \cong B\Sigma_3$ and we have already noticed that $\{B\Sigma_3\} = 1$. We then look at the stratum V_1 where exactly one coordinate is zero. Again H acts transitively so that $[V_1/H]$ is isomorphic to the classifying stack of the stabiliser of $(1, 1, 0)$ say. This stabiliser is isomorphic to $\Sigma_2 \times \mathbf{G}_m$ so that $\{[V_1/H]\} = \{B\Sigma_2\}\{B\mathbf{G}_m\}$. We have however shown that $\{B\Sigma_2\} = 1$. Similarly, for V_2 the locus where exactly two coordinates are zero we get $[V_2/H] \cong B\mathbf{G}_m^2$ and we finally

get a contribution $\{BH\}$ from $[\{0\}/H]$. Moving over this last term to the left hand side of $\{[V/H]\} = \mathbb{L}^3\{BH\}$ we get a formula for $(\mathbb{L}^3 - 1)\{BH\}$ and as $\mathbb{L}^3 - 1$ is invertible in $K_0(\mathrm{Stck_k})$ we get a formula for $\{BH\}$, more precisely we get that $\{BH\} = (\mathbb{L} - 1)^3$.

- For U we need to consider three distinct lines two of which passes through a fixed point. The analysis is altogether similar to the previous one.

- We have one stratum consisting of a line and a smooth quadric (resp. with a singular point at a fixed point). We can place the line in standard position and compare the space of non-singular quadrics (in the second case passing through a fixed point) with the linear space of all quadrics. The space of singular quadrics is then easily analysed.

- We have the case of a double line and another distinct line and a triple line. The analysis is somewhat simplified by the fact that we work in characteristic different from 2 and 3 which means that the map $\ell \mapsto \ell^2$ from lines to quadrics is an immersion (and similarly for cubes).

One may indeed go through all the calculations to get an explicit formula for $\{\mathcal{M}\}$. One could also argue as follows: We can check quite easily that each contribution to the result is a rational function in $\mathbb{L}$ in $K_0(\mathrm{Stck_k}) = K_0(\mathrm{Spc_k})'$. We can the use the fact that χ_c is injective on such rational functions, indeed it is clear that if $\Sigma_{i \leq N} a_i q^i = 0$ where the a_i are integers and $q = \chi_c(\mathbb{L})$ then $a_i = 0$, and hence it is enough to determine $\chi_c(\{\mathcal{M}\}) \in \overline{K}_0^{\mathrm{pol}}(\mathrm{Coh_k})$ as a rational function in $\chi_c(\mathbb{L})$. By Theorem 2.5 this can be done by taking the ordinary Euler characteristic with compact support. Furthermore, $\mathcal{M}$ is a smooth stack so the cohomology of compact support is dual to the ordinary cohomology. Now, if $\mathcal{M}_{0,1}$ is the stack of smooth proper genus 1 curves with a distinguished point we have a map $\mathcal{M}_{0,1} \to \mathcal{M}$ given by $(E, p) \mapsto (E, \mathcal{O}(3p))$. It is clear that it induces an isomorphism on cohomology with characteristic zero coefficients. Hence, we get that $\chi_c(\{\mathcal{M}\}) = \chi_c(\{\mathcal{M}_{0,1}\}) = \chi_c(\mathbb{L})$ and we conclude that $\{\mathcal{M}\} = \mathbb{L}$.

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